

Exercise List: Properties and examples of convexity and smoothness

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Time to get familiarized with convexity, smoothness and a bit of strong convexity.

Notation: For every $x, y, \in \mathbb{R}^d$ let $\langle x, y \rangle \stackrel{\text{def}}{=} x^\top y$ and let $\|x\|_2 = \sqrt{\langle x, x \rangle}$.

Let $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ be the smallest and largest singular values of A defined by

$$\sigma_{\min}(A) \stackrel{\text{def}}{=} \min_{x \in \mathbb{R}^d} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \sigma_{\max}(A) \stackrel{\text{def}}{=} \max_{x \in \mathbb{R}^d} \frac{\|Ax\|_2}{\|x\|_2}. \quad (1)$$

Thus clearly

$$\frac{\|Ax\|_2^2}{\|x\|_2^2} \leq \sigma_{\max}(A)^2, \quad \forall x \in \mathbb{R}^d. \quad (2)$$

Let $\|A\|_F^2 \stackrel{\text{def}}{=} \text{Tr}(A^\top A)$ denote the Frobenius norm of A . Finally, a result you will need, for every symmetric matrix G the $L2$ induced matrix norm can be equivalently defined by

$$\|G\|_2 = \sigma_{\max}(G) = \sup_{x \in \mathbb{R}^d, x \neq 0} \frac{|\langle Gx, x \rangle|}{\|x\|_2^2} = \max_{x \in \mathbb{R}^d, x \neq 0} \frac{\|Gx\|_2}{\|x\|_2}. \quad (3)$$

1 Convexity

We say that a twice differentiable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in \mathbb{R}^d, \lambda \in [0, 1]. \quad (4)$$

or equivalently

$$v^\top \nabla^2 f(x) v \geq 0, \quad \forall x, v \in \mathbb{R}^d. \quad (5)$$

We say that f is μ -strongly convex if

$$v^\top \nabla^2 f(x) v \geq \mu \|v\|_2^2, \quad \forall x, v \in \mathbb{R}^d. \quad (6)$$

Ex. 1 — We say that $\|\cdot\| \rightarrow \mathbb{R}_+$ is a norm over \mathbb{R}^d if it satisfies the following three properties

1. **Point separating:** $\|x\| = 0 \Leftrightarrow x = 0, \forall x \in \mathbb{R}^d$.
2. **Subadditive:** $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in \mathbb{R}^d$
3. **Homogeneous:** $\|ax\| = |a|\|x\|, \forall x \in \mathbb{R}^d, a \in \mathbb{R}$.

Part I

Prove that $x \mapsto \|x\|$ is a convex function.

Part II

For every convex function $f : y \in \mathbb{R}^m \mapsto f(y)$, prove that $g : x \in \mathbb{R}^d \mapsto f(Ax - b)$ is a convex function, where $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$.

Part III

Let $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ be convex for $i = 1, \dots, m$. Prove that $\sum_{i=1}^m f_i$ is convex.

Part IV

For given scalars $y_i \in \mathbb{R}$ and vectors $a_i \in \mathbb{R}^d$ for $i = 1, \dots, m$ prove that the *logistic regression* function $f(x) = \sum_{i=1}^m \ln(1 + e^{-y_i \langle x, a_i \rangle})$ is convex.

Part V

Let $A \in \mathbb{R}^{m \times d}$ have full column rank. Prove that $f(x) = \frac{1}{2} \|Ax - b\|_2^2$ is $\sigma_{\min}^2(A)$ -strongly convex.

Hint 1: Use the formulation (6) of strong convexity

2 Smoothness

We say that a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L -smooth if

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad (9)$$

or equivalently if f is twice differentiable then

$$v^\top \nabla^2 f(x) v \leq L\|v\|_2^2, \quad \forall x, v \in \mathbb{R}^d. \quad (10)$$

Ex. 2 — *Part I*

Prove that $x \mapsto \frac{1}{2}\|x\|^2$ is 1-smooth.

Part II

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be twice differentiable and L -smooth. Show that

$$\sigma_{\max}(\nabla^2 f(x)) = \|\nabla^2 f(x)\|_2 \leq L.$$

Part III

For every twice differentiable L -smooth function $f : y \in \mathbb{R}^m \mapsto f(y)$, prove that $g : x \in \mathbb{R}^d \mapsto f(Ax - b)$ is a smooth function, where $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$. Find the smoothness constant of g .

Part IV

Let $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ be a twice differentiable and L_i -smooth for $i = 1, \dots, m$. Prove that $\frac{1}{n} \sum_{i=1}^m f_i$ is $\sum_{i=1}^m \frac{L_i}{n}$ -smooth.

Part V

For given scalars $y_i \in \mathbb{R}$ and vectors $a_i \in \mathbb{R}^d$ for $i = 1, \dots, m$ prove that the *logistic regression* function $f(x) = \frac{1}{m} \sum_{i=1}^m \ln(1 + e^{-y_i \langle x, a_i \rangle})$ is smooth. Find the smoothness constant!

Part VI

Let $A \in \mathbb{R}^{m \times d}$ be any matrix. Prove that $\|Ax - b\|_2^2$ is $\sigma_{\max}^2(A)$ -smooth.

Hint 1: Use the formulation (10) of smoothness