Billiards. Exercises

June 28, 2017

1. Consider two point-masses on a half-line with $m_1 = 1, m_2 = 10^8$. The maximal number of collisions in this system is 31415. Explain the relation to π.

2. Find a 6-periodic billiard trajectory in every right triangle and interpret it as a periodic motion of two mass points on a segment, subject to elastic collisions.

3. Consider the motion of three mass points m_1, m_2, m_3 on a circle, subject to elastic collisions. Assume that the center of mass of the points has zero angular speed. Prove that this is the billiard in an acute triangle with the angles
$$\tan^{-1}\left(m_i\sqrt{\frac{m_1+m_2+m_3}{m_1m_2m_3}}\right),\ i=1,2,3.$$

4. Prove that the foot points of the altitudes of an acute triangle form therein a 3-periodic billiard trajectory (Fagnano trajectory).

5. In which angles can the billiard reflection be continuously defined for the trajectories that hit the corner?

6. Consider the elastic collision of two identical balls in $\mathbb{R}^3$, one at rest and the other moving. Show that after collision they will move in orthogonal directions.

7. Consider two identical discs of radius r on the torus $\mathbb{R}^2/\mathbb{Z}^2$, with a fixed center of mass and subject to elastic collisions. Describe the motion of this system as a 2-dimensional billiard.

8. Deduce Snell's Law from Fermat's Principle.

9. Let the speed of light in the upper half plane be $v(x, y) = y$. Prove that the trajectories of light are half-circles with the centers on the x-axis and the vertical rays (the upper half-plane model of hyperbolic geometry).

10. (i) A (plane) periscope is a system of two mirrors that send a (say) vertical beam of light to a vertical beam, inducing an invertible transformation of one beam to another (that is, a transformation of their normal sections).

Given an arbitrary local transformation of segments, show that there exists a periscope that realizes it.
(ii)* Which local transformation of 2-dimensional disc can be realized by a periscope in $\mathbb{R}^3$?

11. Prove that a smooth convex plane body has at least two diameters. What about dimension 3? Dimension n?

12. Prove that the set of 2-periodic billiard trajectories has zero area (with respect to the area form ω).

13. What is the effect of refraction (subject to Snell's Law) on ω?

14. What is the topology of the space of non-oriented lines in the plane? Of non-oriented great circles on the sphere?

15. How does the change of origin affect the coordinates on the space of oriented lines?

16. (i) Prove that, up to a factor, ω is the only isometry-invariant area form on the space of oriented lines.
(ii) Does the space of oriented lines in $\mathbb{R}^2$ have an isometry-invariant metric? What about the space of oriented great circles on the sphere S^2?

17. Let Γ be a closed convex plane curve, and γ a closed, possibly self-intersecting, curve inside Γ; let L and ℓ be their lengths. Prove that there exists a line that intersects γ at least $[2\ell/L]$ times.

18. The distance between the lines on a ruler paper is 1. Find the probability that a needle of length 1, randomly dropped on the paper, intersects a line. What is the expected number of intersections for a needle of length L? An arbitrary curve of length L? (Buffon's needle problem).

19. (i) Formulate and prove Crofton's formula for S^2. Apply it to prove that the total curvature of a closed space curve is not less than 2π.
(ii)* Prove that the total curvature of a knotted curve in $\mathbb{R}^3$ is not less than 4π (Fáry-Milnor theorem).

20. Find the average area of a plane projection of a unit cube.

21. Define the length of a rectangular box as the sum of its dimensions. Can a box of smaller length contain a box of greater length? Consider 2- and 3-dimensional versions of the problem.

22. Prove that the geometric and analytic definitions of confocal conics are equivalent.

23. Prove that a billiard trajectory in an ellipse that intersect the segment between the foci remains tangent to a confocal hyperbola.

24. Prove that the geometric and analytic formulations of integrability of the billiard inside an ellipse are equivalent.

25. Prove that the product of distances from the foci of an ellipse to a segment of the billiard trajectory is an integral of the billiard ball map.

26. Consider an ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
and consider the diagonal linear map that takes it to a confocal ellipse. Show that the points related by this map lie on a confocal hyperbola.

27. Let F_1 and F_2 be the foci of an ellipse. The billiard reflection gives a transformation of the oriented lines through F_1 to the oriented lines through F_2. Identify pencils of lines with the projective line $\mathbb{RP}^1$ via the stereographic projection. Show that the resulting transformation of $\mathbb{RP}^1$ is Möbius (fractional-linear). Deduce that the billiard trajectory through the foci tends to the great axis of the ellipse.

28. Find a geometric proof of 'the most elementary theorem of elementary geometry'.

29. Prove the Euler-Fuss relations for $n = 3$ and $n = 4$:
$$\frac{1}{R-a} + \frac{1}{R+a} = \frac{1}{r}; \quad \frac{1}{(R-a)^2} + \frac{1}{(R+a)^2} = \frac{1}{r^2}$$
where $R > r$ are the radii of the circles and a is the distance between their centers.

30. Prove that a pair of nested ellipses can be taken to a pair of circles by a suitable projective transformation.