The Obstacle Problem.

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Contents

1 Introduction.

The point of this course is to give you some exposure to advanced theory for partial differential equations and to introduce regularity theory for free boundaries. Modern theory of PDE is based on a variety of techniques from advanced analysis - in particular measure theory, Sobolev spaces and functional analysis. This means that normally one need to take at least three advanced courses in addition to an advanced PDE course before one can begin to comprehend modern PDE theory. This is unfortunate since, as so often in mathematics, many of the ideas are elementary. These notes are an attempt to introduce some modern PDE theory in an as gentle way as possible - mostly through first year undergraduate analysis.

The structure of the notes are as follows; in section 2 we show existence of solutions to the obstacle problem through te direct method in the calculus of variations. Since the direct method is based on convergence of functions with control only over the integrability of the derivatives we need Sobolev spaces and some functional analysis. The main ideas of Sobolev space theory and the minimal knowledge of functional analysis is introduced in an appendix to section 2. Most roofs are supplied, if at all, for the one dimensional case - which should give the general idea of why the theory works. In section 3 we apply the direct method to the obstacle problem.

In section 4 we develop the weak regularity theory for the obstacle problem and in section 5 the strong regularity theory. The last three sections will be dedicated to the regularity of the free boundary. In section 6 we will talk about the measure theoretic properties of free boundaries, without proofs. Section 7 will try to characterize continuously differentiable functions through a reverse Taylor Theorem. And in the final section we will use the reverse Taylor Theorem to prove regularity of the free boundary at flat points.

The proof of free boundary regularity in section 8 has not appeared in print before. The method is much more robust than the traditional free boundary regularity proofs and can be generalized to many new situations. Some of these generalizations have appeared, or will shortly appear, in print. For the purposes of a survey course it seamed reasonable to write out the proof of the simplest case in some detail.

The prerequisites for reading these notes are minimal. One needs to have some knowledge of the classical theory of harmonic functions. Most of the results on Sobolev spaces requires knowledge of integration theory - but I believe that it should be possible to understand the main ideas without it. In section 6 we will also refer, without proof, to some results from the theory of sets of finite perimeter, but section 6 is included as background and not necessary to understand for the rest of these notes.

Reading instructions for Spring mini School on Nonlinear PDE and Free Boundary Problems, Yerevan 29th-31st March: We will focus the lectures on section 7-8. To really get the idea one need have a basic understanding of Sobolev spaces and the $W^{2,2}$ estimates from Theorem 4.2, which we will briefly cover in the first lecture. But in order to intrduce one of the main protagonists in our story, weak convergence, we will breafly talk about existence of minimizers (chapter 2) in the first lecture.

The magic, the mathematical magic, happens in the entire development of the theory, not just what we will cover during the lectures. Starting from the variational formulation of the problem; which has a perfect balance between flexibility (we do not demand to much of the solutions - then we would not be able to prove existence) and structure (the solutions we do get have enough good properties) for us in order to build a viable theory. The underpinning of the Sobolev spaces, which generalize the undergraduate calculus in a way that gives us a Bolzano-Weierstrass theorem for weak convergence but still preserve enough of the classical properties of the derivative. I wish that we had time to talk about it all in the leisurely phase that a respectful treatment of the material would demand. But we will not have the time; therefore I leave you with these notes instead in the hope that some will read them in their entirety, and think deep and hard about them, only then can a true appreciation of the theory develop.

Notation: We will use the letter $\mathcal{D}$ to denote a domain in $\mathbb{R}^n$ (n will always denote the space dimension) - that is $\mathcal{D}$ is an open set. Throughout these notes $\mathcal{D}$ will be bounded and connected. The topological boundary of $\mathcal{D}$ will be denoted $\partial\mathcal{D}$ and the outward pointing normal of $\mathcal{D}$ will be denoted ν. An open ball of radius r with center x^0 will be denoted $B_r(x^0)$. The upper half-ball will be denoted $B_r^+(x^0) = \{x \in B_r(x^0);\ x_n > 0\}$. By diam($\mathcal{D}$) we mean the diameter of the set $\mathcal{D}$ - that is by definition the diameter of the smallest ball that contains the set $\mathcal{D}$.

We will denote points in space by $x = (x_1, x_2, ...x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, ..., z_n)$ et.c. We often think of these vectors to be variables. Fixed points are often denoted by a superscript x^0, y^0 et.c. At times we will use a prime, as in x', to denote the first $n-1$ components in a vector: $x' = (x_1, x_2, ..., x_{n-1})$. In a slight abuse of notation we will at times interpret x' as a vector in $\mathbb{R}^n$ with n:th component equal to zero $x' = (x_1, x_2, ..., x_{n-1}, 0)$ and at times we will also write $(x', t) = (x_1, x_2, .., x_{n-1}, t)$; in particular $x = (x', x_n)$. We believe that it it will be clear from context what we intend.

We will use several function spaces in these notes. By $C(\mathcal{D})$ we mean all the continuous functions in the domain $\mathcal{D}$, by $C^{0,\alpha}(\mathcal{D})$ we mean the space of all continuous functions, $u(x)$, on $\mathcal{D}$ such that $|u(x) - u(y)| \le C|x-y|^\alpha$ equipped with the norm $\|u\|_{C^{0,\alpha}(\mathcal{D})} = \sup_{x\in\mathcal{D}} |u(x)| + \sup_{x,y\in\mathcal{D}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$. If $\alpha \in (0,1)$ we often write $C^\alpha(\mathcal{D}) = C^{0,\alpha}(\mathcal{D})$. Similarly we write $C^{k,\alpha}(\mathcal{D})$ for all k times continuously differentiable functions $u(x)$ defined on $\mathcal{D}$ such that all k :th order derivatives of $u(x)$ belong to $C^\alpha(\mathcal{D})$: that is $D^k u(x) \in C^\alpha(\mathcal{D})$. We will allow α to be zero and then just write $C^{k,\alpha}(\mathcal{D}) = C^k(\mathcal{D})$.

We will use the spaces $L^2(\mathcal{D})$ for all integrable functions on $\mathcal{D}$ such that $\|u\|^2_{L^2(\mathcal{D})} = \int_{\mathcal{D}} |u|^2 dx < \infty$. We will also use the Sobolev space $W^{1,2}(\mathcal{D})$ for all functions $u(x)$ defined on $\mathcal{D}$ such that both $u(x)$ and $\nabla u(x)$ are integrable and $\|u\|^2_{W^{1,2}(\mathcal{D})} = \|u\|^2_{L^2(\mathcal{D})} + \|\nabla u\|^2_{L^2(\mathcal{D})} < \infty$. Similarly we will use $W^{k,2}(\mathcal{D})$ for the space of functions such that all derivatives of order up to k belong to $L^2(\mathcal{D})$.

We will use a sub-script $C_c(\mathcal{D})$, $C_c^2(\mathcal{D})$ et.c. to denote the functions $u \in C_c(\mathcal{D})$, $u \in C_c^2(\mathcal{D})$ et.c with compact support. And $W^{k,2}(\mathcal{D})$ for functions that are identically equal to zero (in the trace sense) on $\partial\mathcal{D}$.

We will use ∇ for the gradient operator $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, ..., \frac{\partial u(x)}{\partial x_n}\right)$ and Δ for the Laplace operator $\Delta u(x) = \sum_{j=1}^n \frac{\partial^2 u(x)}{\partial x_j^2}$.

2 The Calculus of Variations.

The calculus of variations consists in finding, and describing properties, of functions that minimize some energy. To be specific we look for a function $u(x)$ that minimizes the following energy

$$J(u) =_{\mathrm{df}} \int_{\mathcal{D}} F(\nabla u(x))dx \tag{2.1}$$

among all functions in[1]

$$K = \{u \in W^{1,2}(\mathcal{D}),\ u(x) = f(x) \text{ on } \partial\mathcal{D}\}. \tag{2.2}$$

In physics the energy is usually some combination of several energies, for instance potential and kinetic energy. Calculus of variations is very important for applications. In these notes we will be interested in the mathematical theory.

The main problem in the calculus of variations is to show existence of minimizers to the minimization problem (2.1) in the set K. It is easy to construct examples of functionals for which no minimizers exists. The easiest example of a minimization problem for which no minima exists is for discontinuous functions defined on $\mathbb{R}$, which has nothing to do with Sobolev spaces.

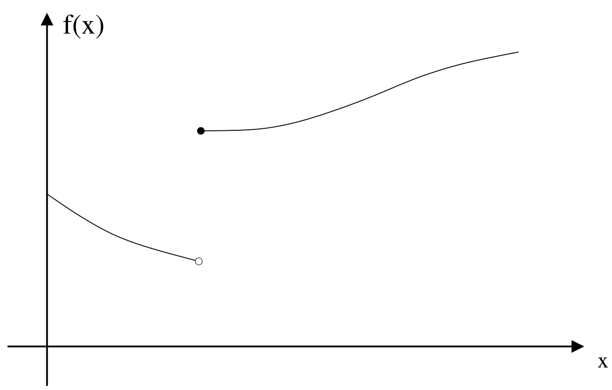

Figure 1: *A one dimensional example where the minimum does not exist.*

The example in figure 1 clearly shows that we need some assumptions in order to assure that minimizers for a given minimization problem exist. In order to understand the existence properties for the problem described by (2.1)-(2.2) we begin by stating a simple Theorem that we understand on minimization of functions in $\mathbb{R}^n$:

Theorem 2.1. *If $f(x)$ is a lower semi-continuous[2] function on a closed and bounded set $K \subset R^n$ and $f(x) > -\infty$ then $f(x)$ achieves its minimum in K.*

[1] Here we use the notation $W^{1,2}(\mathcal{D})$ which is the set of all functions such that $\int_{\mathcal{D}}(|\nabla u|^2 + |u|^2) < \infty$. This is a Sobolev space. We refer to the appendix to this section for more details on Sobolev spaces.

[2] Remember that $f(x)$ is lower semi-continuous if $f(x_0) = \liminf_{x\to x_0} f(x)$

Proof: The proof is, as we already know, done in several simple steps.

1. $V_f = \{f(x);\ x \in K\}$ is bounded from below which implies, by the completeness property of the real numbers, that $\inf_{x\in K} f(x)$ exists.
2. We may thus find $x^j \in K$ s.t. $\lim_{j\to\infty} f(x^j) = \inf_{x\in K} f(x)$.
3. Since K is a compact set in $\mathbb{R}^n$ the Bolzano-Weierstrass Theorem implies that there exists a convergent sub-sequence of x^j which we will denote $x^{j_k} \to x^0 \in K$.
4. Lower semi-continuity of f implies that
$$f(x^0) \le \lim_{k\to\infty} f(x^{j_k}) = \inf_{x\in K} f(x).$$
Clearly by the definition of infimum $f(x_0) \ge \inf_{x\in K} f(x)$. It follows that $f(x_0) = \inf_{x\in K} f(x)$; thus $f(x)$ achieves its minimum in x^0.

□

We would like to replicate this theorem in the more complicated setting of the minimization of the functional (2.1) in the set (2.2). The main difference for the minimization problem ((2.1)-(2.2)):

$$\text{minimize } J(u) = \int_{\mathcal{D}} F(\nabla u)dx \quad u \in K = \{W^{1,p}(\mathcal{D});\ u = f \text{ on } \partial\mathcal{D}\}.$$

is that K is no longer finite dimensional which implies that we no longer have a Bolzano-Weierstrass compactness Theorem.

But we may (almost) replicate the strategy of Theorem 2.1

1. To show that the minimum is a well defined number we just need to assume that $F(\nabla u) \ge -C$. That would imply that $J(u) \ge -C\int_{\mathcal{D}} dx = -C|\mathcal{D}|$; then the existence of a minimum of the functional exists by the completeness property of the real numbers. Notice that this does not imply that there exists a function $u \in K$ such that $J(u) = \inf_{u\in K} J(u)$.
2. By the property that an infimum exists we can find a sequence $u^j \in K$ s.t. $\lim_{j\to\infty} J(u^j) = \inf_{u\in K} J(u)$.
3. The space $W^{1,2}$ is weakly compact[3] so if
$$\|u^j\|_{W^{1,2}(\mathcal{D})} \le C \tag{2.3}$$
then, there exists a sub-sequence u^{k_j} such that, $u^{k_j} \rightharpoonup u^0 \in W^{1,2}(\mathcal{D})$.
In order to assure (2.3) we need to assume that the functional is **(Coercive)**:
$$J(u) \to \infty \text{ as } \|u\|_{W^{1,2}}(\mathcal{D}) \to \infty. \tag{2.4}$$
Clearly (2.4) implies that $\|u^j\|_{W^{1,2}(\mathcal{D})}$ is bounded if $J(u^j)$ is bounded, which it certainly is if $J(u^j) \to \inf_{u\in K} J(u) \in \mathbb{R}$.

[3]See the appendix to this chapter for a brief explanation of this property.

4. We need to show that $J(u)$ is lower semi-continuous with respect to weak convergence in $W^{1,2}(\mathcal{D})$.

In the above strategy there is no real problem with the first three points. We can clearly decide whether the first and third points holds if we have an explicit functional $J(u) = \int_{\mathcal{D}} F(\nabla u)dx$ - at least we can easily imagine classes of functionals where the first and third point holds. The second point is just a simple fact that follows from the definition of the infimum.

The fourth point needs some further comment. In general, it is not meaningful to have theorems if we cannot verify when the assumptions are satisfied. It would therefore be much more reassuring if we could find some criteria that implies lower semi-continuity for the functional. This is what we will do next. As so often in mathematics we will try to understand a complicated situation by constructing an example easy enough for us to explicitly calculate it. In PDE theory that usually means construction a one dimensional example since the power of one dimensional calculus allows us to do most calculations explicitly in one dimension.

Example: Consider the one-dimensional minimizing problem

$$\text{minimize } J_F(f(x)) = \int_0^1 F(f'(x))dx \tag{2.5}$$

in the set

$$K = \left\{ f \in W^{1,2}(0,1);\ f(0) = 0 \text{ and } f(1) = 1 \right\}.$$

We need to choose our function $F(\cdot)$ which we choose quite randomly to be the function with the graph

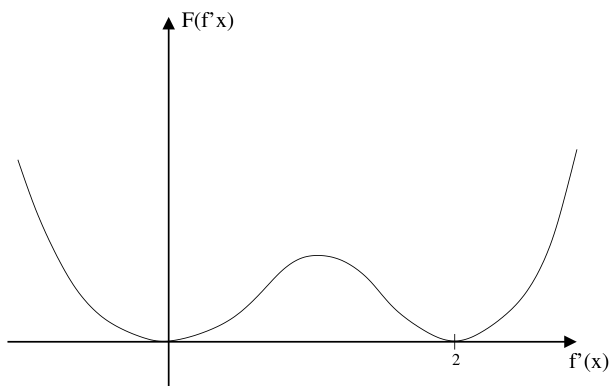

Figure 2: *The graph of the function $F(\cdot)$.*

Since $F(\cdot) \geq 0$ we can conclude that $J(f) \geq 0$ for all functions $f \in K$. But if

$$f'(x) = \begin{cases} 0 \text{ if } x \in A \\ 2 \text{ if } x \notin A \end{cases} \tag{2.6}$$

for some set A then the energy $J_F(f'(x)) = 0$ since $F(0) = F(2) = 0$. Thus any function $f(x)$ of the form (2.6) will be a minimizer to (2.5). Notice that such a minimizer can arbitrarily well approximate (in $C^0([0,1])-$norm) any function $g(x)$ satisfying $0 \leq g' \leq 2$. This can be clearly seen in the following picture:

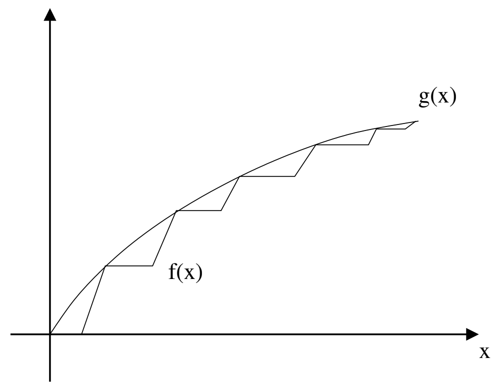

Figure 3: *Graphic representation of how a function $f(x)$ whose derivative takes the values* 0 *and* 2 *approximates an arbitrary function $g(x)$ with derivative* $0 \leq g'(x) \leq 2$.

This implies that for any function $g(x) \in K$ such that $0 \leq g'(x) \leq 2$ we can find a sequence $f^j \in K$ such that $f^j \to g$ uniformly and $J(f^j(x)) = 0$. But $J(g(x))$ may very well be strictly positive, for instance if $g(x) = x$. Thus the functional $J_F(f)$ defined in (2.5) is not lower semi-continuous.[4]

The question we need to ask is: *Is the problem that the function F is zero at two different points?* A simple example shows that that is not the case.

Consider for instance the one dimensional minimization problem

$$\text{minimize } J_G(f(x)) = \int_0^1 G(f'(x))dx$$

in the set

$$K = \left\{ f \in W^{1,p}(0,1);\ f(0) = 0 \text{ and } f(1) = 1 \right\},$$

where the function G is given by the graph:

[4]Notice that the lower semi-continuity of J_F is not really related to the continuity of F. We may very well, as in the example, have that F is a continuous function but J_F is not continuous on the space $W^{1,2}(\mathcal{D})$.

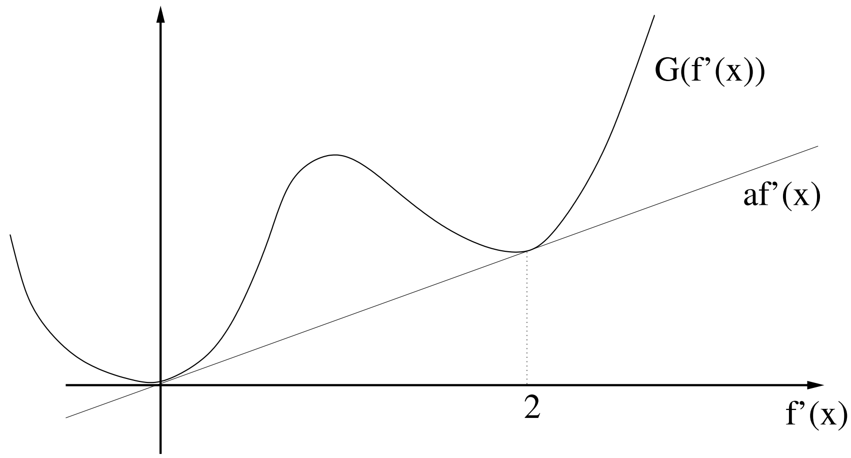

Figure 4: *The graph of the function $G(f'(x))$ and $af'(x)$.*

If we subtract the linear function $af'(x)$ from $G(f'(x))$ we will get a function with graph looking like the one in Figure 3; we may even assume that $G(f'(x)) = F(f'(x)) + af'(x)$. This leads to

$$J_G(f'(x)) = \int_0^1 G(f'(x))dx = \int_0^1 F(f'(x))dx + a\int_0^1 f'(x)dx =$$

$$= J_F(f'(x)) + af(1) - af(0),$$

where we used an integration by parts in the last equality. Since $af(1)-af(0) = a$ for all $f \in K$ we can conclude that

$$J_G(f'(x)) = J_F(f'(x)) + a \text{ for all } f \in K.$$

And since J_F and J_G only differ by a constant we can conclude that J_G cannot have a minimizer since J_F does not have a minimizer. □

What conclusion can we draw from this example? The reason that there a minimizer to $\int_0^1 G(f'(x))dx$ does not exist was that we could touch the graph of G from below, at two different points, by a linear function. That is: G is not convex. Clearly convexity is a necessary condition for a minimizer to exist, at least for minimization in $\mathbb{R}$. It turns out that convexity is the assumption needed in any dimension $\mathbb{R}^n$, not just for examples in $\mathbb{R}$, for the functional $J(u)$ to be lower semi-continuous.[5] We are ready to formulate and prove an existence theorem for minimizers.

Theorem 2.2. *Assume that $\mathcal{D}$ is a given bounded domain[6] and that $F : \mathbb{R}^n \mapsto \mathbb{R}$ is a continuously differentiable function satisfying:*

[5] Here we are talking about minimization for scalar valued functions $u(x)$. If $u(x)$ is vector valued there exists many different notions of convexity (quasi-convexity, poly-convexity etc.c.) that implies existence in different situations. We will not consider minimization problems with vector valued functions in this course.

[6] By a domain we mean an open set in $\mathbb{R}^n$.

1. *There exists a constant C such that $F(\nabla u) \geq -C$*
2. $\int_{\mathcal{D}} F(\nabla u) \to \infty$ *as* $\|\nabla u\|_{L^2(\mathcal{D})} \to \infty$
3. *$F(p)$ is convex:*

$$F(\boldsymbol{q}) \geq F(\boldsymbol{p}) + F'(\boldsymbol{p})(\boldsymbol{q} - \boldsymbol{p}) \quad \text{for any } \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}.$$

Then for any closed (under weak limits) and non-empty sub-set $K \subset W^{1,2}(\mathcal{D})$ there exists a function $u \in K$ such that

$$\int_{\mathcal{D}} F(\nabla u) dx = \inf_{v \in K} \int_{\mathcal{D}} F(\nabla v) dx.$$

Proof: The proof follows the same steps as the proof in the one dimensional case.

Since $F(\nabla u) \geq -C$ we can conclude that for any $u \in K$

$$J(u) = \int_{\mathcal{D}} F(\nabla u(x)) dx \geq -C \int_{\mathcal{D}} dx = -C|\mathcal{D}|,$$

where $|\mathcal{D}|$ denotes the volume of the set $|\mathcal{D}|$. Thus the set of values $J(u)$ can obtain is bounded from below and therefore the the number $m = \inf_{u \in K} J(u)$ exists and is well defined.

We may therefore find a sequence u^j such that $J(u^j) \to m$. Notice that since $J(u) \to \infty$ as $\|\nabla u\|_{L^2(\mathcal{D})} \to \infty$ it follows that $\|\nabla u^j\|_{L^2(\mathcal{D})}$ is bounded. By weak compactness there is a sub-sequence u^{j_k} and a function $u^0 \in W^{1,2}(\mathcal{D})$ such that $u^{j_k} \rightharpoonup u^0$ in $W^{1,2}(\mathcal{D})$. Since K is closed it follows that $u^0 \in K$. There is no loss of generality to assume that the sub-sequence u^{j_k} is the full sequence u^j.

We need to verify that $J(u^0) = m$. To that end we calculate as $j \to \infty$

$$\begin{aligned} m \leftarrow \int_{\mathcal{D}} F(\nabla u^j) dx = \int_{\mathcal{D}} F(\nabla u^0 + \nabla(u^j - u^0)) dx \geq \{\text{covexity}\} \geq \\ \geq \int_{\mathcal{D}} F(\nabla u^0) + \int_{\mathcal{D}} \underbrace{F'(\nabla u^0) \cdot (\nabla u^j}_{\to F'(\nabla u^0) \cdot \nabla u^0} - \nabla u^0) dx \to \\ \to \int_{\mathcal{D}} F(\nabla u^0) + \int_{\mathcal{D}} F'(\nabla u^0) \cdot (\nabla u^0 - \nabla u^0) dx = \int_{\mathcal{D}} F(\nabla u^0) dx. \end{aligned} \tag{2.7}$$

The calculation (2.7) proves that $J(u^0) = m = \inf_{u \in K} J(u)$. □

Proposition 2.1. *If the function $F(p)$ in Theorem 2.2 is strictly convex*

$$F(\boldsymbol{q}) > F(\boldsymbol{p}) + F'(\boldsymbol{p})(\boldsymbol{q} - \boldsymbol{p}) \quad \text{for any } \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R},$$

for all $\mathbf{p} \neq \mathbf{q}$, the domain $\mathcal{D}$ is connected and the set K convex. Then the minimizer is unique up to additive constants. In particular the minimizer is unique among all functions with the same boundary data.

Proof: Assume that we have two minimizers $u(x) \in K$ and $v(x) \in K$ then by strict convexity and that both minimizers have the same energy

$$0 = \int_{\mathcal{D}} F(\nabla u(x))dx - \int_{\mathcal{D}} F(\nabla v(x))dx \geq \int_{\mathcal{D}} F'(\nabla u) \cdot (\nabla(v-u))dx, \quad (2.8)$$

with equality only if $\nabla u(x) = \nabla v(x)$.

Since u is a minimizer and K is convex, which implies that $u(x) + t(v(x) - u(x)) \in K$ for $t \in [0,1]$,

$$0 \leq \int_{\mathcal{D}} F(\nabla u(x) + t\nabla(v(x) - u(x)))dx. \quad (2.9)$$

Dividing by $t > 0$ and letting $t \to 0$ in (2.9) gives

$$0 \leq \lim_{t\to 0^+} \frac{1}{t} \int_{\mathcal{D}} F(\nabla u(x) + t\nabla(v(x) - u(x)))dx = \int_{\mathcal{D}} F'(\nabla u) \cdot (\nabla(v-u))dx.$$

Comparing this to (2.8) we see that we must have equality in (2.8). But as we remarked earlier we only have equality in (2.8) if $\nabla u(x) = \nabla v(x)$. It follows that $u(x) - v(x)$ is constant.

If $u(x) = v(x)$ on $\partial\mathcal{D}$ then clearly $u(x) - v(x) = 0$ and it follows that the minimizer is unique among the functions with the same boundary data. □

The importance of the existence theorem and the uniqueness proposition is, of course, the following Theorem from L.C. Evans' *Partial Differential Equations*.

Theorem 2.3. *[Dirichlet's Principle.] Assume that* $u \in C^2(\overline{\mathcal{D}})$ *solves*

$$\begin{aligned} \Delta u(x) &= f(x) && \text{in } \mathcal{D} \\ u(x) &= g(x) && \text{on } \partial\mathcal{D}. \end{aligned} \quad (2.10)$$

Then

$$\int_{\mathcal{D}} \left(\frac{1}{2}|\nabla u(x)|^2 - u(x)f(x) \right) dx = \inf_{w\in K} \int_{\mathcal{D}} \left(\frac{1}{2}|\nabla w(x)|^2 - w(x)f(x) \right) dx, \quad (2.11)$$

where $K = \{w \in C^2(\overline{\mathcal{D}});\ w(x) = g(x) \text{ on } \partial\mathcal{D}\}$.

Conversely, if $u \in K$ *satisfies (2.11) then* $u(x)$ *solves (2.10).*

The minimizer $u(x)$ of the Dirichlet energy therefore is a very good candidate to be the solution to Laplace equation - and is indeed the unique solution if it is C^2. It is indeed the case that the minimizer of the Dirichlet energy is C^2 and we thus have a new way to find solutions to Laplace equation. Furthermore, the existence proofs in elementary PDE theory[7] only works for very nice domains, such as balls and half-spaces, where the explicit calculations can be carried out. The variational existence theorem works for any domain. However, one has to

[7]Such as Fourier series expansion of the solution or explicitly constructing a Green's function.

impose some restrictions on the domain in order to assure that the minimizer assumes the right boundary data, see Corollary 2.2.

Exercises.

1. [THE MAXIMUM PRINCIPLE.] You already know that if $u(x)$ is harmonic in $\mathcal{D}$ then $\sup_{x\in\mathcal{D}} u(x) = \sup_{x\in\partial\mathcal{D}} u(x)$. Prove this using that harmonic functions minimize the Dirichlet energy $\int_{\mathcal{D}} |\nabla u(x)|^2 dx$ among all functions with the same boundary values.

   HINT: *Let* $M = \sup_{x\in\partial\mathcal{D}} u(x)$ *and consider the energy of the function* $u_M(x) = \min(u(x), M,)$. *Show that* u_M *has less than or equal Dirichlet energy as* u *and use uniqueness of minimizers.*

2. * [COMPARISON PRINCIPLE.] Assume that $u(x)$ and $v(x)$ are harmonic in $\mathcal{D}$ and that $u(x) \leq v(x)$ on $\partial\mathcal{D}$. Use the variational formulation of the Dirichlet problem to prove that $u(x) \leq v(x)$ in $\mathcal{D}$.

   HINT: *See previous exercise.*

3. [GENERALIZATIONS.] Theorem 2.3 states that finding minimizers to the Dirichlet energy is the same as solving $\Delta u(x) = 0$. However, the real strength of the calculus of variations is that it easily generalizes to a wide variety of problems.

   (a) Assume that $a(x) > 0$ and $a(x) \in C^1(\mathcal{D})$. Show that there exists a minimizer to $\int_{\mathcal{D}} a(x)|\nabla u(x)|^2 dx$ in the set

   $$K = \{u \in W^{1,2}(\mathcal{D});\ u(x) = f(x) \text{ on } \partial\mathcal{D}\}.$$

   What partial differential equation does the minimizer solve?

   HINT: *Follow the proof of Theorem 2.3.*

   (b) Assume that we have a minimizer[8] to $\int_{\mathcal{D}} |\nabla u(x)|^p$ in

   $$K = \{u \in W^{1,p}(\mathcal{D});\ u(x) = f(x) \text{ on } \partial\mathcal{D}\}$$

   for some $1 < p < \infty$. Assume furthermore that the minimizer $u(x) \in C^2(\mathcal{D})$. What partial differential equation does $u(x)$ solve?[9]

   (c) Finally, consider a minimizer $u(x)$ to $\int_{\mathcal{D}} |\Delta u(x)|^2 dx$. What PDE does $u(x)$ solve?[10]

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[8] This is actually just as easy to prove as the theorem above. At least if one knows a little basic functional analysis that I do not feel that we have time for right now.

[9] The PDE is called the $p-$laplacian and the function $u(x)$ is called $p-$harmonic. This example is interesting since the $p-$laplacian is non-linear and one can not construct solutions using the Green-function methods used in Evans to find solutions.

[10] This is an important example in $\mathbb{R}^2$ since it is a model for the bending of a thin metal plate.

4. [NEUMANN DATA.] Let $u(x)$ minimize the Dirichlet energy $\int_{B_1(0)} |\nabla u(x)|^2 dx$ in the set

$$K = \{u \in W^{1,2}(B_1(0));\ u(x) = f(x) \text{ on } \partial B_1(0) \cap \{x_n \leq 0\}\}.$$

   Notice that the boundary data is only imposed on the negative half of the ball. Show that

$$\frac{\partial u(x)}{\partial \nu} = 0 \text{ on } \partial B_1(0) \cap \{x_n > 0\},$$

   where $\nu = \frac{x}{|x|}$ is the outer normal of $\partial B_1(0)$.

   HINT: *Make variations $w(x) = u(x) + t\phi(x)$ in the proof of Theorem 2.3 where $\phi(x)$ is not necessarily zero on $\partial B_1(0) \cap \{x_n > 0\}$.*

5. ** Find an $F \in C^1(\mathbb{R}^n \mapsto \mathbb{R})$ so that the functional $J(u) = \int_{\mathcal{D}} F(\nabla u(x))dx$ admits several minimizers.

   HINT: *What is the difference in the assumptions in the existence Theorem 2.2 and the uniqueness Proposition 2.1?*

2.1 Appendix. A painfully brief introduction to Sobolev spaces and weak convergence.

In this appendix we gather some facts about the convergence of functions that we need in order to show the existence of minimizers. Ideally we would prove all the results in the appendix - but we will not strive for the ideal. Some of the results belong properly to functional analysis and measure theory and would take us to far off topic. We will however try to motivate some of the results informally.

The central concept in analysis is convergence. We see this already in a first calculus course; when we learn about the convergence of difference quotients in order to define derivatives and the convergence of Riemann sums in order to define the integral. The next order of sophistication is to consider the convergence of functions.

What does it mean for a sequence of functions f_j to converge to a function f_0? The answer to that question is manifold and it depends much on the situation which kind of convergence is relevant. In the calculus of variations we are often interested in sequences of functions $u^j(x)$ such that

$$\int_{\mathcal{D}} |\nabla u^j(x)|^2 dx \leq C$$

for some given domain $\mathcal{D}$ and constant C. That is functions whose derivative is square integrable. The natural space to consider would therefore be

Definition 2.1. *Let $\mathcal{D} \subset \mathbb{R}^n$ be a given set. Then we write $L^2(\mathcal{D})$ for the set of all functions $u(x)$ defined on $\mathcal{D}$ such that*

$$\|u\|_{L^2(\mathcal{D})} = \left(\int_{\mathcal{D}} |u(x)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

Remark: *The L in $L^2(\mathcal{D})$ stands for Lebesgue since for these spaces are always considering the Lebesgue integral. However, if you are not familiar with the Lebesgue integral you can think of the integral as being a Riemann integral. There are some instances where we will use a fact that is true only for the Lebesgue integral but not for the Riemann integral - you will have to accept those facts.*

The importance of the space $L^2(\mathcal{D})$ comes form the following Theorem.

Theorem 2.4. *The Space $L^2(\mathcal{D})$ with the norm $\|u\|_{L^2(\mathcal{D})}$ is a complete space. That is if $u^j \in L^2(\mathcal{D})$ is a Cauchy sequence, $\lim_{j,k\to\infty} \|u^j - u^k\|_{L^2(\mathcal{D})} = 0$, then there exists a function $u^0 \in L^2(\mathcal{D})$ such that $\lim_{j\to\infty} \|u^j - u^0\|_{L^2(\mathcal{D})} = 0$.*

Remark: *The completeness is not true for Riemann integrable functions. For instance the sequence $u^j(x) \in L^2(0,1)$ defined so that $u^j(x) = 1$ if x is one of the first j rational numbers (in some ordering) and $u^j(x) = 0$ else. Then the Riemann integral $\int_0^1 |u^j|^2 dx = 0$ and u^j converges point-wise to a function $u^0(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ that is not Riemann integrable.*

In $L^2(\mathcal{D})$ we say that $u^j(x) \to u^0(x)$ if $\|u^j - u^0\|_{L^2(\mathcal{D})} \to 0$ as $j \to \infty$. We would want this convergence to have some good properties. We would in particular like the Bolzano-Weierstrass Theorem[11] to hold. Let us briefly remind ourselves of the Bolzano-Weierstrass Theorem in $\mathbb{R}^n$.[12]

Theorem 2.5. *Let $K \subset \mathbb{R}^n$ be a closed and bounded set and $\{x^j\}_{j=1}^\infty$ be a sequence such that $x^j \in K$. Then there exists a sub-sequence $\{x^{j_k}\}_{k=1}^\infty$ of $\{x^j\}_{j=1}^\infty$ such that $\lim_{k\to\infty} x^{j_k}$ exists.*

Sketch of the Proof: Let $x^j = (x_1^j, x_2^j, ..., x_n^j)$ then by the Bolzano-Weierstrass Theorem in $\mathbb{R}$ we may find a sub-sequence, which we may denote $\{x_1^{j_{k,1}}\}_{k=1}^\infty$, of $\{x_1^j\}_{j=1}^\infty$ such that $\{x_1^{j_{k,1}}\}_{k=1}^\infty$ converges.

Again by the one dimensional Bolzano-Weierstrass Theorem we may find a subsequence of $j_{k,1}$, lets denote it $j_{k,2}$, such that $\{x_2^{j_{k,2}}\}_{k=1}^\infty$ converges. We may then find a subsequence of $j_{k,2}$, lets denote it $j_{k,3}$, such that $\{x_3^{j_{k,3}}\}_{k=1}^\infty$ converges et.c.

[11] That is every sequence u^j such that $\|u^j\|_{L^2(\mathcal{D})} \leq C$ for some constant C has a convergent sub-sequence.

[12] I am a strong believer that there is no difference between first year calculus courses and PhD level courses and must therefore consistently refer back to undergraduate stuff in all my courses. But you shouldn't complain - I refer to PhD level stuff in my first year undergraduate courses as well so you are better off than my first year students...

In the end we find a sequence $j_k = j_{k,n}$ such that $\{x_n^{j_k}\}_{k=1}^{\infty}$ converges. But since j_k, by construction, is a sub-sequence of each sequence $\{j_{k,l}\}_{k=1}^{\infty}$, $l = 1, 2, ..., n$ it follows that

$$\lim_{k\to\infty} x_l^{j_k} = x_l^0 \quad \text{for} \quad l = 1, 2, ..., n.$$

This finishes the proof. □

In order to gain some feeling for the convergence properties of functions in $L^2(\mathcal{D})$ we need to make some calculations. Before we start our investigation into convergence in L^2 we remind ourselves of Paresval's Theorem and the Cauchy-Schwartz inequality.

Theorem 2.6. *Let $u(x), v(x) \in L^2(-\pi, \pi)$ and*

$$u(x) = \frac{a_0(u)}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k(u)\frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k(u)\frac{\sin(kx)}{\sqrt{\pi}}$$

and

$$v(x) = \frac{a_0(v)}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k(v)\frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k(v)\frac{\sin(kx)}{\sqrt{\pi}}$$

then

- [Parseval's Theorem.]

$$\int_{-\pi}^{\pi} |u(x)|^2 dx = a_0(u)^2 + \sum_{k=1}^{\infty} \left(a_k(u)^2 + b_k(u)^2\right).$$

- [Cauchy-Schwartz Inequality.] *For any two functions $g, h \in L^2(\mathcal{D})$ the following inequality holds*

$$\int_{\mathcal{D}} g(x)h(x)dx \le \left(\int_{\mathcal{D}} |g(x)|^2 dx\right)^{1/2} \left(\int_{\mathcal{D}} |h(x)|^2 dx\right)^{1/2} \tag{2.12}$$

this can be formulates for $u(x)$ and $v(x)$ as

$$\int_{-\pi}^{\pi} u(x)v(x)dx = a_0(u)a_0(v) + \sum_{k=1}^{\infty} \left(a_k(u)a_k(v) + b_k(u)b_k(v)\right).$$

Parseval's Theorem allows us to view $L^2(-\pi, \pi)$ as an infinite dimensional vector space with basis $\frac{\sin(kx)}{\sqrt{\pi}}$ and $\frac{\cos(kx)}{\sqrt{\pi}}$. This allows us to make some of the calculations more explicit in the one dimensional setting.[13] Next we provide

[13] Any basic course in functional analysis will teach you that $L^2(\mathcal{D})$ is a Hilbert space and thus has a basis. The difference with $L^2(-\pi, \pi)$ is that we may write down the basis explicitly with familiar trigonometric functions.

an example that shows that the Bolzano-Weierstrass Theorem does not hold in $L^2(-\pi,\pi)$.

Example: *The Bolzano-Weierstrass Theorem does not hold on* $L^2([-\pi,\pi])$. *In particular, we may find a sequence of functions* $u^j \in L^2([-\pi,\pi])$ *such that* $\|u^j\|_{L^2([-\pi,\pi])} = 1$ *but* u^j *does not contain any convergent sub-sequence.*

Proof of the example: We will use some Fourier analysis. For a function $u(x)$ we will write

$$u(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k \frac{\sin(kx)}{\sqrt{\pi}},$$

where

$$a_k = \int_{-\pi}^{\pi} u(x) \frac{\cos(kx)}{\sqrt{\pi}} dx \quad \text{and} \quad b_k = \int_{-\pi}^{\pi} u(x) \frac{\sin(kx)}{\sqrt{\pi}} dx.$$

The sequence $u^j(x) = \frac{1}{\sqrt{\pi}}\cos(jx)$ will satisfy $\|u^j\|_{L^2(-\pi,\pi)} = 1$. The only non-zero Fourier-coefficient of u^j is $a_j = 1$. We claim that $u^j(x)$ cannot have any convergent subsequence. To see this we assume the contrary; that $u^j \to u^0$, for a sub-sequence, where

$$u^0(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k \frac{\sin(kx)}{\sqrt{\pi}}.$$

By Parseval's Theorem

$$\int_{-\pi}^{\pi} |u^j(x) - u^0(x)|^2 dx = \left(a_0^2 + \sum_{k=1,k\neq j}^{\infty} \left(a_k^2 + b_k^2\right) + (1-a_j)^2 + b_j^2 \right)$$

$$\geq \left(a_0^2 + \sum_{k=1}^{j-1} \left(a_k^2 + b_k^2\right) \right).$$

Since, if $u^j \to u^0$, then the right hand side tends to zero (for some sub-sequence) we can conclude that if $u^k \to u^0$ then $a_k = 0$ and $b_k = 0$ for all k. That is $u^0 = 0$. But this implies that

$$\|u^j - u^0\|_{L^2(-\pi,\pi)} = \|u^j\|_{L^2(-\pi,\pi)} = 1$$

which contradicts $\|u^j - u^0\|_{L^2(-\pi,\pi)} \to 0$. We can conclude that $u^j \not\to u^0$ for any subsequence. □

The above example shows that we cannot hope for a Bolzano-Weierstrass Theorem for $L^2(\mathcal{D})$ for any domain $\mathcal{D}$.

However, we are still able to salvage something from the proof of the finite dimensional Bolzano-Weierstrass Theorem to the infinite dimensional case. If we assume that $u^j(x)$ is a sequence of functions in $L^2(-\pi,\pi)$ with the Fourier expansion

$$u^j(x) = \frac{a_0^j}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} a_k^j \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{\infty} b_k^j \frac{\sin(kx)}{\sqrt{\pi}}$$

and $\|u^j\|_{L^2(-\pi,\pi)} \leq C$ then the sequence of numbers $\{c_k^j\}_{j=1}^{\infty}$ ($c_k^1 = a_k^1, c_k^2 = b_k^1, c_k^3 = a_k^2, c_k^4 = b_k^2, c_k^5 = a_k^2 ...$) must be bounded: $|c_k^j| \leq C$ and $|c_k^j| \leq C$. This means that we can find a sub-sequence $c_1^{l_{1,j}}$ of c_1^j that converges $c_1^{l_{1,j}} \to c_1^0$, a subsequence $l_{2,j}$ of $l_{1,j}$ such that $c_2^{l_{2,j}} \to a_2^0$ and inductively a subsequence $l_{k,j}$ of $l_{k-1,j}$ such that $c_k^{l_{k,j}} \to c_k^0$.

By choosing the diagonal sequence $l_j = l_{j,j}$, just as in the Arzela-Ascoli Theorem, we see that $a_k^{l_j} \to a_k^0$ and $b_k^{l_j} \to b_k^0$ for any k. Thus there exists a subsequence u^{l_j} whose Fourier coefficients all converge to a_k^0 and b_k^0 respectively. We may thus find a subsequence of any bounded sequence $u^j \in L^2(-\pi,\pi)$ such that all the Fourier coefficients converge - it is not what we want but it will have to do.

The mode of convergence of the sequence above is called weak convergence. Since weak convergence is a much more general concept than something that applies only for $L^2(-\pi,\pi)$ we will give the classical definition of weak convergence. Later we will prove that the above *convergence of all the Fourier coefficients* is indeed weak convergence defined as follows.

Definition 2.2. *We say that u^j converges weakly in $L^2(\mathcal{D})$ to u^0, or simply write $u^j \rightharpoonup u^0$, if for every function $v \in L^2(\mathcal{D})$*

$$\int_{\mathcal{D}} u^j(x)v(x)dx \to \int_{\mathcal{D}} u^0(x)v(x)dx. \tag{2.13}$$

Remark: *Observe that we use the symbol $\rightharpoonup$ and not $\to$ for weak convergence. It is also important that we have the same function $v(x)$ in the integrals (2.13).*

Our first Lemma for weakly converging functions is.

Lemma 2.1. *Let $u^j \rightharpoonup u^0$ in $L^2(\mathcal{D})$ for some domain $\mathcal{D}$. Then*

$$\liminf_{j\to\infty} \|u^j\|_{L^2(\mathcal{D})} \geq \|u^0\|_{L^2(\mathcal{D})}.$$

Proof: Consider

$$\liminf_{j\to\infty}\left(\|u^j\|_{L^2(\mathcal{D})} - \|u^0\|^2_{L^2(\mathcal{D})}\right) = \liminf_{j\to\infty}\int_{\mathcal{D}}\left(|u^j(x)|^2 - |u^0(x)|^2\right)dx =$$

$$= \liminf_{j\to\infty}\int_{\mathcal{D}}\left(|u^j(x)|^2 - |u^0(x)|^2 - 2u^j(x)u^0(x) + 2|u^0(x)|^2\right)dx = \tag{2.14}$$

$$= \liminf_{j\to\infty}\int_{\mathcal{D}}\left(|u^j(x) - u^0(x)|^2\right)dx \geq 0,$$

where we used that, by weak convergence,

$$\lim_{j\to\infty}\int_{\mathcal{D}}\left(u^j(x)u^0(x)-|u^0(x)|^2\right)dx=\int_{\mathcal{D}}\left(u^0(x)u^0(x)-|u^0(x)|^2\right)dx=0,$$

in the equality leading to (2.14). The Lemma follows. □

So far we have not shown that *any sequence whatsoever* of functions $u^j \in L^2(\mathcal{D})$ converges weakly - but we suspect that the convergence of the Fourier coefficients should imply weak convergence. But as a matter of fact we regain the Bolzano-Weierstrass Theorem if we consider weak convergence instead of strong.

Theorem 2.7. [Weak Compactness Theorem.] *Let $u^j(x) \in L^2(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$, be a bounded sequence*[14]*. Then there exists a sub-sequence, which we still denote u^j, that converges weakly $u^j \rightharpoonup u^0$ for some function $u^0 \in L^2(\mathcal{D})$.*

Sketch of proof: The Theorem is formulated for any domain $\mathcal{D} \subset \mathbb{R}^n$. We will for the sake of simplicity sketch the proof in $L^2(-\pi,\pi)$. The general proof is based on the same ideas - but would use more functional analysis that we neither want to prove not presuppose for this course.

Step 1: *The set up.*

We assume that u^j is a sequence of functions with the Fourier expansions

$$u^j(x)=\frac{a_0^j}{2}+\sum_{k=1}^{\infty}a_k^j\cos(kx)+\sum_{k=1}^{\infty}b_k^j\sin(kx).$$

Since, by assumption, $\|u^j\|_{L^2(-\pi,\pi)}$ is bounded it follows that $|a_k^j|$ and $|b_k^j|$ are bounded. By the one dimensional Bolzano-Weierstrass Theorem we can find a subsequence, $u^{j_{0,k}}$ of u^j such that $a_0^{j_{0,k}} \to a_0^0$ and $b_0^{j_{0,k}} \to b_0^0$. Choosing a subsequence again, which we denote $u^{j_{1,k}}$, we can assure that $a_1^{j_{1,k}} \to a_1^0$ and $b_1^{j_{1,k}} \to b_1^0$. Continuing inductively we may define the diagonal sequence $j_{k,k}$ for which $a_l^{j_{k,k}} \to a_l^0$ and $b_l^{j_{k,k}} \to b_l^0$ for every l.

We will denote

$$u^0(x)=\frac{a_0^0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}a_k^0\frac{\cos(kx)}{\sqrt{\pi}}+\sum_{k=1}^{\infty}b_k^0\frac{\sin(kx)}{\sqrt{\pi}}.$$

We claim that $u^{j_{k,k}} \rightharpoonup u^0$. In order to simplify notation we will write $j = j_{k,k}$ knowing that we have choses a sub-sequence already.

We need to show (2.13) for any function $v \in L^2(-\pi,\pi)$. We define the Fourier-coefficients of v according to

$$v(x)=\frac{a_0^v}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}a_k^v\frac{\cos(kx)}{\sqrt{\pi}}+\sum_{k=1}^{\infty}b_k^v\frac{\sin(kx)}{\sqrt{\pi}}.$$

[14] When we talk about bounded sequences in L^2 we always mean sequences with bounded norm: $\|u^j\|_{L^2(-\pi,\pi)} \le C_u$ for some constant C_u independent of j.

Step 2: *Proof if $a_k^v = 0$ and $b_k^v = 0$ if $k > M$.*

Proof of Step 2: By Pareval's Theorem we get

$$\int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v(x)dx = \sum_{k=0}^{\infty}(a_k^j - a_k^0)a_k^v + \sum_{k=1}^{\infty}(b_k^j - b_k^0)b_k^v = \tag{2.15}$$

$$= \left\{ \begin{array}{l} \text{use } a_k^v, b_k^v = 0 \\ \text{for } k > M \end{array} \right\} = \sum_{k=0}^{M}(a_k^j - a_k^0)a_k^v + \sum_{k=1}^{M}(b_k^j - b_k^0)b_k^v \to 0$$

where we used that $a_k^j - a_k^0 \to 0$ and $b_k^j - b_k^0 \to 0$. This implies (2.13).

Step 3: *Proof for general $v \in L^2(-\pi, \pi)$.*

Proof of Step 3: We may write $v(x) = v_M(x) + w_M(x)$ where

$$v_M(x) = \frac{a_0^v}{\sqrt{2\pi}} + \sum_{k=1}^{M} a_k^v \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=1}^{M} b_k^v \frac{\sin(kx)}{\sqrt{\pi}}$$

and

$$w_M = \sum_{k=M+1}^{\infty} a_k^v \frac{\cos(kx)}{\sqrt{\pi}} + \sum_{k=M+1}^{\infty} b_k^v \frac{\sin(kx)}{\sqrt{\pi}}.$$

Since $v \in L^2(-\pi, \pi)$ we know, from Parseval's Theorem, that the series

$$\sum_{k=1}^{\infty}(a_k^v)^2 + \sum_{k=1}^{\infty}(b_k^v)^2$$

converges. We may therefore, for any $\epsilon > 0$, choose M so that

$$\|w_M\|_{L^2(-\pi,\pi)}^2 = \sum_{k=M+1}^{\infty} (a_k^v)^2 + \sum_{k=M+1}^{\infty} (b_k^v)^2 < \epsilon^2. \tag{2.16}$$

It follows that

$$\left| \int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v(x)dx \right| =$$

$$= \left| \int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v_M(x)dx \right| + \left| \int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) w_M(x)dx \right| \leq \tag{2.17}$$

$$\leq \left| \underbrace{\int_{-\pi}^{\pi} \left(u^j(x) - u^0(x)\right) v_M(x)dx}_{\to 0 \text{ by Step 2}} \right| + \|u^j - u^0\|_{L^2(-\pi,\pi)} \|w_M\|_{L^2(-\pi,\pi)},$$

where we used Cauchy-Schwartz inequality in the last step of the calculation.

Notice that by the triangle inequality and Lemma 2.1

$$\|u^j - u^0\|_{L^2(-\pi,\pi)} \leq \|u^j\|_{L^2(-\pi,\pi)} + \|u^0\|_{L^2(-\pi,\pi)} \leq 2C_u \tag{2.18}$$

where C_u is the bound on $\|u^j\|_{L^2(-\pi,\pi)}$.

By choosing M large enough we can, by (2.16) and (2.18), make the term

$$\|u^j - u^0\|_{L^2(-\pi,\pi)}\|w_M\|_{L^2(-\pi,\pi)} < \epsilon. \tag{2.19}$$

Passing to the limit in (2.17) and using (2.19) we can conclude that

$$\lim_{j\to\infty}\left|\int_{-\pi}^{\pi}\left(u^j(x) - u^0(x)\right)v(x)dx\right| < \epsilon,$$

for any $\epsilon > 0$. The Theorem follows. □

When we do calculus of variations we are really interested in functions whose derivatives are in the space $L^2(\mathcal{D})$. We need the following definitions.

Definition 2.3. *Let u be integrable in some domain $\mathcal{D} \subset \mathbb{R}^n$. Then if there exists an integrable function $w(x)$ such that*

$$\int_{\mathcal{D}} \frac{\partial v(x)}{\partial x_i} u(x)dx = -\int_{\mathcal{D}} v(x)w(x)dx \text{ for every } v(x) \in C_c^1(\mathcal{D}) \tag{2.20}$$

then we say that $u(x)$ is weakly differentiable in x_i and that $w(x)$ is the weak x_i-derivative of $u(x)$ and write $\frac{\partial u(x)}{\partial x_i} = w(x)$.

Remark: *Notice that the definition is made so as the partial integration formula works. In particular, if $u(x)$ is weakly differentiable in x_i then (2.20) become the normal integration by parts formula*

$$\int_{\mathcal{D}} \frac{\partial v(x)}{\partial x_i} u(x)dx = -\int_{\mathcal{D}} v(x)\frac{\partial u(x)}{\partial x_i}dx.$$

It follows directly that every continuously differentiable function is weakly differentiable.

Definition 2.4. *Let $u(x) \in L^2(\mathcal{D})$ be weakly differentiable in every direction x_i, $i = 1, 2, ..., n$ and the weak derivatives $\frac{\partial u}{\partial x_i} \in L^2(\mathcal{D})$ for all $i = 1, 2, ..., n$. Then we say that $u \in W^{1,2}(\mathcal{D})$. We call the space of all such functions equipped with the norm*

$$\|u\|_{W^{1,2}(\mathcal{D})} = \left(\int_{\mathcal{D}} |u(x)|^2 dx + \int_{\mathcal{D}} |\nabla u(x)|^2 dx\right)^{1/2}, \tag{2.21}$$

where $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, ..., \frac{\partial u(x)}{\partial x_n}\right)$.

We will also write $W^{k,2}(\mathcal{D})$ for all functions defined on $\mathcal{D}$ such that all weak derivatives up to order k exists and

$$\|u\|_{W^{k,2}(\mathcal{D})} = \left(\sum_{|\alpha|\le k}\int_{\mathcal{D}} |D^\alpha u|^2\right)^{1/2} < \infty,$$

where the summation is over all mutiindexes α of length $|\alpha| \le k$.

Remark: *Often in analysis one uses the Sobolev space $W^{k,p}(\mathcal{D})$ with norm:*

$$\|u\|_{W^{k,p}(\mathcal{D})} = \left(\sum_{|\alpha|\le k} \int_{\mathcal{D}} |D^\alpha u|^p \right)^{1/p} < \infty.$$

We will not use this space in this course.

Lemma 2.2. *The space $W^{1,2}(\mathcal{D})$ with norm (2.21) is a complete space.*

Furthermore every bounded sequence of functions $u^j \in W^{1,2}(\mathcal{D})$ has a subsequence u^{j_k} so that $u^{j_k} \rightharpoonup u^0$ and $\frac{\partial u^{j_k}}{\partial x_i} \rightharpoonup \frac{\partial u^0}{\partial x_i}$ in $L^2(\mathcal{D})$ for some $u^0 \in W^{1,2}(\mathcal{D})$.

Remark: *We say that the subsequence u^{j_k} converges weakly to u^0 in $W^{1,2}(\mathcal{D})$, written $u^{j_k} \rightharpoonup u^0$ in $W^{1,2(\mathcal{D})}$.*

Proof: That $W^{1,2}(\mathcal{D})$ is complete follows from the same statement for $L^2(\mathcal{D})$, Theorem 2.4.

By Theorem 2.7 we can clearly extract a subsequence such that u^{j_k} and $\frac{\partial u^{j_k}}{\partial x_i}$, for all $i = 1, 2, ..., n$, converges weakly. We need to show that the limit of the partial derivatives converges to the partial derivatives of the limit $u^{j_k} \rightharpoonup u^0$. That is easy. For any $\phi \in C^1(\mathcal{D})$ we have

$$\int_{\mathcal{D}} \frac{\partial \phi(x)}{\partial x_i} u^0(x)dx \leftarrow \int_{\mathcal{D}} \frac{\partial \phi(x)}{\partial x_i} u^{j_k}(x)dx = \tag{2.22}$$

$$-\int_{\mathcal{D}} \phi(x) \frac{\partial u^{j_k}(x)}{\partial x_i} dx \rightarrow -\int_{\mathcal{D}} \phi(x) \lim_{j_k\to\infty} \frac{\partial u^{j_k}(x)}{\partial x_i} dx,$$

Since (2.22) holds for every ϕ it follows that $\frac{\partial u^{j_k}(x)}{\partial x_i} \rightharpoonup \frac{\partial u^0(x)}{\partial x_i}$. This finishes the proof. □

Weak derivatives are less intuitive. But they are also more flexible. One can, for instance, prove the following lemma.

Lemma 2.3. *Assume that $u, v \in W^{1,2}(\mathcal{D})$, then $\max(u, v) \in W^{1,2}(\mathcal{D})$. In particular for any constant c, $\max(u, c) \in W^{1,2}(\mathcal{D})$.*

Observe that the lemma is manifestly not true for functions $u \in C^1(\mathcal{D})$ so weakly differentiable functions are malleable.

We need one final, and rather subtle, concept regarding Sobolev spaces in order to use them in the calculus of variations.

Theorem 2.8. [Traces.] *Let $\mathcal{D}$ be a bounded domain with continuously differentiable boundary. Then there exists an operator*

$$T : W^{1,2}(\mathcal{D}) \mapsto L^2(\partial\mathcal{D})$$

that assigns boundary values (in the trace sense) of $u \in W^{1,2}(\mathcal{D})$ onto the boundary $\partial\mathcal{D}$.

Furthermore $Tu = u\lfloor_{\partial\mathcal{D}}$ for all functions $u \in C(\overline{\mathcal{D}}) \cap W^{1,2}(\mathcal{D})$.

Before we prove this theorem we will try to motivate its importance. We do not require functions $u \in W^{1,2}(\mathcal{D})$ to be continuous - and assume even less we that functions $u \in W^{1,2}(\mathcal{D})$ have a continuous extension to $\overline{\mathcal{D}}$. It is therefore not obvious that we can talk about boundary values for functions in $W^{1,2}(\mathcal{D})$. Consider the simple example $\cos(1/x) \in C(0,1)$ which is continuous, but have no continuous extension to $[0,1)$ - wherefore we can not in any meaningful way ascribe a boundary value to $\cos(1/x)$. In order to solve the Dirichlet problem we need the function to take a prescribed boundary value; and therefore we need to have a notion of boundary values. The *"boundary values"* are given by the trace operator T whose existence is assured by the Theorem.

Sketch of the proof of Theorem 2.8: This proof actually goes beyond this course. But I want to indicate how the Trace Theorem is proved for several reasons. First of all we need the theorem. Secondly, in proving the theorem we will encounter some standard techniques in PDE theory. Thirdly, the proof will also indicate why there is more to Sobolev spaces than an integration by parts formula. In particular, I want to stress that measure theory (as in *"Advanced Real Analysis"*) is important for the general analysis of PDE.

Step 1 [Straightening of the boundary.]: *It is enough to prove that the operator* $T : W^{1,2}(B_1^+(0)) \mapsto L^2(B_{3/4}(0) \cap \{x_n = 0\})$, *where* $B_1^+(0) = B_1(0) \cap \{x_n > 0\}$ exists.

Proof of Step 1: By definition a domain has C^1 boundary if we can cover $\partial B_1(0)$ by finitely many balls $B_{r_j}(x^j)$ such that $\partial\mathcal{D} \cap B_{2r_j}(x^j)$ is the graph of a C^1 function f^j in some coordinate system.

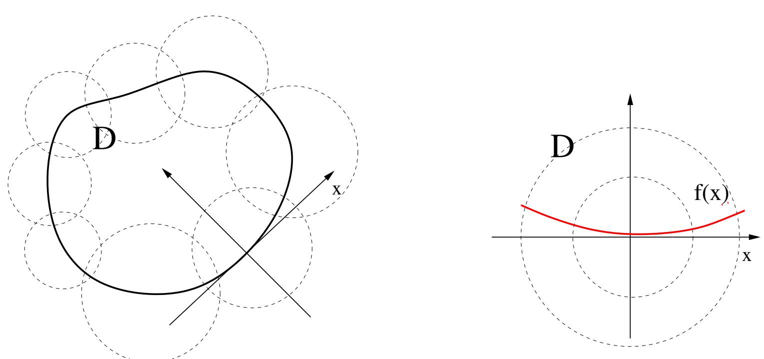


Figure: *The left figure shows a domain $\mathcal{D}$ with C^1 boundary. This means that we may cover $\partial\mathcal{D}$ by a finite number of balls $B_{r_j}(x^j)$ such that for each ball there is a coordinate system so that $\partial\mathcal{D} \cap B_{2r_j}(x^j)$ is a graph in the coordinate system. The right picture shows the same coordinate system rotated and the boundary portion $\partial\mathcal{D} \cap B_{2r_j}(x^j)$ (in red) which is clearly the graph of some function $f(x)$. We will change coordinates to straighten the red part of the boundary.*

The idea of the proof is that we may *"straighten the boundary"* in $B_{2r_j}(x)$ by defining the new coordinates $\hat{x}$ so that

$$(x_1, x_2, ..., x_n) = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n + f^j(\hat{x}')) \\ \Leftrightarrow \\ (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) = (x_1, x_2, ..., x_n - f^j(x')).$$

Then the part of the boundary $x_n = f^j(x')$ will be mapped to the hyperplane $\hat{x}_n = 0$ in the $\hat{x}$ coordinates. We may write the function $u(x)$ in these coordinates as $\hat{u}(\hat{x})$.

By the chain rule we get that

$$\frac{\partial \hat{u}(\hat{x})}{\partial \hat{x}_i} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial \hat{x}_i} \frac{\partial u(x)}{\partial x_k} = \sum_{k=1}^{n} \left(\frac{\partial u(x)}{\partial x_k} + \frac{\partial f^j(x')}{\partial x_k} \frac{\partial u}{\partial x_n} \right).$$

In particular, $|\nabla \hat{u}(\hat{x})|$ will be comparable in size with $|\nabla u(x)|$.

Since $f(x)$ is continuously differentiable we can conclude that

$$\int_{\mathcal{D}\cap B_{2r}(x^0)} |\nabla \hat{u}(\hat{x})|^2 d\hat{x} \leq C \int_{\mathcal{D}\cap B_{2r}(x^0)} |\nabla u(x)|^2 dx,$$

where the constant C only depend on the maximum value of $|\nabla' f(x')|$.

Notice that $\hat{u}(\hat{x})$ is defined in a set where part of the boundary is straight (in the $\hat{x}$ coordinates). If we can define boundary values for $\hat{u}$ on the straight part of the boundary then we can define boundary values of $u(x)$ on the portion of the boundary that lays in $B_{r_j}(x^j)$ by the equality $u(x) = \hat{u}(x', 0)$. But the entire boundary $\partial\mathcal{D}$ can be covered by finitely many balls $R_{r_j}(x^j)$ so we can define boundary values for $u(x)$ on the entire boundary $\partial\mathcal{D}$.

Step 2: *Let* $u(x) \in W^{1,2}(B_2^+(0))$ *then*

$$\int_{B_1'(0)} |u(x',t) - u(x',s)|^2 dx' \leq |s-t| \|\nabla u(x)\|^2_{L^2(B_2^+)}$$

where $B_1'(0) = \{x' \in \mathbb{R}^{n-1}; |x'| \leq 1\}$ *is the unit ball in the* x' *coordinates.*

Proof of step 2: Using the fundamental theorem of calculus[15]

$$\int_{B_1'(0)} |u(x',t) - u(x',s)|^2 dx' = \int_{B_1'(0)} \left| \int_s^t \frac{\partial u(x)}{\partial x_n} dx_n \right|^2 dx' \leq$$

$$\leq |s-t| \int_{B_1'(0)} \int_s^t \left| \frac{\partial u(x)}{\partial x_n} \right| dx' dx_n \leq |s-t| \|\nabla u(x)\|^2_{L^2(B_2^+)}$$

where we used the Cauchy-Schwartz inequality (2.12)[16] in the first inequality.

[15]This is just a sketch of a proof. But this is the part of the proof that is most sketchy. We have note proved, nor will we prove, that the fundamental theorem of calculus is applicable to functions in $W^{1,2}$ in the way we use it here.

[16]With $g(x) = \frac{\partial u(x)}{\partial x_n}$ and $h(x) = 1$.

Step 3: *Let $u(x) \in W^{1,2}(B_2^+(0))$ then the limit $\lim_{t\to 0'} u(x',t) = u^0(x',0)$ exists (and is therefore unique) and the function $u^0(x',0) \in L^2(B_1'(0))$ and satsifies the estimate*

$$\|u^0\|_{L^2(B_1'(0))} \leq C\|u\|_{W^{1,2}(B_2^+(0))}$$

where the constant C does not depend on u. This finishes the proof.

Proof of Step 3: Since $u(x) \in W^{1,2}(B_2^+(0))$ it follows that

$$\int_0^{1/4}\int_{B_1'(0)} |u(x)|^2 dx' dx_n \leq \int_{B_2^+(0)} |u(x)|^2 dx < \infty, \tag{2.23}$$

where we used $B_1'(0)\times(0,1/4) \subset B_2^+(0)$ in the first inequality and the definition of $W^{1,2}(B_2^+(0))$ (see Definition 2.4) in the second inequality.

From (2.23) we can conclude that there exists an $s \in (0,1/4)$ such that

$$\int_{B_1'(0)} |u(x',s)|^2 dx' \leq 4\int_{B_2^+(0)} |u(x)|^2 dx.$$

This implies that $u(x',s) \in L^2(B_1'(0))$ and therefore, from step 2, that $u(x',t) \in L^2(B_1'(0))$.

By Step 2 the sequence of functions $u(x',s/j)$ will form a Cauchy sequence and is therefore convergent, in $L^2(B_1'(0))$ to some function $u^0(x',0) \in L^2(B_1'(0))$. Also, by step 2,

$$\|u(x',s/j) - u^0(x',0)\|_{L^2(B_1'(0))} \leq \sqrt{s/j}\|\nabla u(x)\|_{L^2(B_2^+)}.$$

We only need to assure that $\lim_{t\to 0^+} u(x',t) = u^0(x',0)$. But that follows from step 2 and the triangle inequality:

$$\|u(x',t) - u^0(x',0)\|_{L^2(B_1'(0))} \leq \tag{2.24}$$

$$\leq \|u(x',t) - u^0(x',s/j)\|_{L^2(B_1'(0))} + \|u(x',s/j) - u^0(x',0)\|_{L^2(B_1'(0))} \leq \tag{2.25}$$

$$\left(\sqrt{t-s/j} + \sqrt{s/j}\right)\|\nabla u(x)\|_{L^2(B_2^+)}, \tag{2.26}$$

where we choose j so large that $s/j \leq t$. It clearly follows from (2.24)-(2.26) that

$$\|u(x',t) - u^0(x',0)\|_{L^2(B_1'(0))} \leq 2\sqrt{t}\|\nabla u(x)\|_{L^2(B_2^+)}. \tag{2.27}$$

It follows that $\lim_{t\to 0'} u(x',t) = u^0(x',0)$ in $L^2(B_1'(0))$. □

We also need two more results on Sobolev spaces and traces.

Corollary 2.1. *Let $\mathcal{D}$ be a bounded C^1 domain and $u \in W^{1,2}(\mathcal{D})$ and $u = f$ on $\partial\mathcal{D}$ in the sense of traces. Then*

$$\|u\|_{L^2(\mathcal{D})} \leq C\left(\|\nabla u\|_{L^2(\mathcal{D})} + \|f\|_{L^2(\partial\mathcal{D})}\right), \tag{2.28}$$

where the constant C does not depend on u.

Sketch of the Proof: We will only sketch the proof. We begin with the special case $f(x) = 0$. Since $\mathcal{D}$ is bounded there is a cube $Q_R = \{x \in \mathbb{R}^n;\ |x_i| < R \text{ for } i = 1, 2, ..., n\}$ and $\mathcal{D} \subset Q_R$. We will extend u to zero in $Q_R \setminus \mathcal{D}$:

$$u(x) = \begin{cases} u(x) & \text{for } x \in \mathcal{D} \\ 0 & \text{for } x \in Q_R \setminus \mathcal{D}. \end{cases}$$

By the fundamental theorem of calculus

$$u(x', x_n) = \int_{-R}^{x_n} \frac{\partial u(x', t)}{\partial x_n} dt. \tag{2.29}$$

If we take absolute values and square both sides of (2.29) and then integrate over $\mathcal{D}$ we get

$$\int_{\mathcal{D}} |u(x)|^2 dx = \int_{\mathcal{D}} \left| \int_{-R}^{x_n} \frac{\partial u(x', t)}{\partial x_n} dt \right|^2 dx \leq$$

$$\leq 2R \int_{\mathcal{D}} \int_{-R}^{x_n} \left| \frac{\partial u(x', t)}{\partial x_n} \right|^2 dt dx \leq 2R \int_{\mathcal{D}} \int_{-R}^{x_n} \left| \nabla u(x', t) \right|^2 dt dx,$$

where we used the Cauchy-Schwartz inequality[17] in the first inequality. Changing the order of integration leads to

$$\int_{\mathcal{D}} |u(x)|^2 dx \leq 2R \int_{-R}^{R} \int_{\mathcal{D}} \left| \nabla u(x', x_n) \right|^2 dx dt \leq 4R^2 \int_{\mathcal{D}} \left| \nabla u(x', x_n) \right|^2 dx,$$

where we also increased the interval of integration from $-R < t < x_n$ to $-R < t < R$ which clearly increases the value of the integral. This proves (2.28) in the case $f(x) = 0$.

For the general case we may define a cut-off function $\psi(x) \in C_c^\infty(\mathcal{D})$ such that $0 \leq \psi(x) \leq 1$ and $\psi(x) = 1$ for $\text{dist}(x, \partial\mathcal{D}) > r_0/2$ where r_0 is the smallest radius in the proof of the Trace Theorem. We then split up $u(x) = (1 - \psi(x))u(x) + \psi(x)u(x)$. Then $\psi(x)u(x)$ has trace equal to zero on $\partial\mathcal{D}$ and the argument in the previous paragraph applies. The function $(1 - \psi(x))u(x)$ can be estimated in terms of the boundary values and the norm $\|\nabla(1 - \psi)u\|_{L^2}$ as in (2.27). We leave the details to the reader. □

The next Corollary states that traces are preserved under week limits in $W^{1,2}(\mathcal{D})$.

Corollary 2.2. *Let $u^j \rightharpoonup u^0$ in $W^{1,2}(\mathcal{D})$ where $\mathcal{D}$ is a bounded C^1 domain. Assume furthermore that $u^j = f$ on $\partial\mathcal{D}$ in the trace sense. Then $u^0 = f$ on $\partial\mathcal{D}$.*

One can also show that the space $W^{1,2}(\mathcal{D})$ is compactly embedded in $L^2(\mathcal{D})$. We will only sketch the main idea of the proof.

---

[17] Theorem 2.6 with $g(x) = \frac{\partial u(x',t)}{\partial x_n}$ and $h(x) = 1$.

Theorem 2.9. [COMPACTNESS OF THE EMBEDDING $W^{1,2} \to L^2$.] *Let $u^j \rightharpoonup u^0$ in $W^{1,2}(\mathcal{D})$ where $\mathcal{D}$ is a bounded C^1-domain. Then $u^j \to u^0$ strongly in $L^2(\mathcal{D})$.*

A sketch of the idea of the proof: We have seen the idea of the proof before. Let us, for simplicity, assume that $\mathcal{D} = Q_R$ and that $u^j = f$ on ∂Q_R for some fixed function f. We need to show that

$$\|u^j - u^0\|^2_{L^2(Q_R)} = \int_{Q_R} |u^j - u^0|^2 dx \to 0.$$

Since $u^j - u^0 \rightharpoonup 0$ in $W^{1,2}(Q_R)$ we know that for almost every $x \in Q_R'$

$$u^j(x) - u^0(x) = \int_0^{x_n} \frac{\partial(u^j(x',t) - u^0(x',t))}{\partial x_n} dt. \tag{2.30}$$

But by weak convergence it follows that, for almost every x',

$$\int_0^{x_n} \frac{\partial(u^j(x',t) - u^0(x',t))}{\partial x_n} dt \to 0.$$

Taking absolute values in (2.30) and integrating over Q_R we get

$$\int_{Q_r} |u^j - u^0|^2 dx = \int_{Q_R} \underbrace{\left| \int_0^{x_n} \frac{\partial(u^j(x',t) - u^0(x',t))}{\partial x_n} dt \right|}_{\to 0 \text{ for a.e. } x} dx \to 0.$$

Writing this I realize that the sketch is to brief to be comprehensible without understanding more about Sobolev spaces than we have covered in these notes. But the main idea is that the function $\nabla(u^j - u^0)$, without absolute values, controls the size of $|u^j(x) - u^0(x)|$ at a.e. point. And since the integral of $\nabla(u^j - u^0)$ will tend to zero if $u^j \rightharpoonup u^0$ in $W^{1,2}$ we can control $|u^j(x) - u^0(x)|$ at almost every point. □

Exercises.

1. Let $\mathcal{D}$ be a bounded domain and $u(x) \in C^1(\mathcal{D})$. Show that $u(x) \in W^{1,2}(\mathcal{D})$.

2. * Consider the function $u(x) = \ln(1/|x|)$ defined in $B_1(0) \subset \mathbb{R}^3$.

   (a) Show that $u(x) \in W^{1,2}(B_1(0))$.

   (b) Conclude that there are discontinuous, and even unbounded, functions $u(x) \in W^{1,2}(B_1(0))$.

3. ** Let $\mathcal{D}$ be a bounded domain and $u(x) \in C^1(\mathcal{D})$.

   (a) If $\mathcal{D} = B_1(0)$ and $u(x) = 0$ on $\partial B_1(0)$ show that

$$u(0) = \frac{1}{\omega_n} \int_{B_1(0)} \frac{y \cdot \nabla u(y)}{|y|^n} dy.$$

HINT: *By the fundamental theorem of calculus*

$$u(0) = -\int_0^1 y \cdot \nabla u(ty)dt$$

for any y such that $|y| = 1$. Integrate this over the unit sphere $\partial B_1(0) = \{y;\ |y = 1|\}$.

(b) Show that for all $x \in B_1(0)$.

$$u(x) = \frac{1}{\omega_n}\int_{B_1(0)} \frac{(y-x)\cdot \nabla u(y)}{|y-x|^n}dy.$$

(c) Use the following inequality, known as Hölder's inequality,

$$\int_{B_1} f(x)g(x)dx \leq \left(\int_{B_1} |f(x)|^p dx\right)^{1/p}\left(\int_{B_1} |f(x)|^q dx\right)^{1/q}$$

for $\frac{1}{p}+\frac{1}{q} = 1$, to show that for any $\epsilon > 0$ there exists a constant C_ϵ such that

$$\sup_{B_1(0)} |u(x)| \leq C_\epsilon \left(\int_{B_1(0)} |\nabla u(x)|^{n+\epsilon}\right)^{\frac{1}{n+\epsilon}}.$$

[REMARK:] *The assumption that $u \in C^1$ is not needed in the above argument. It is enough to assume that $u \in W^{1,n+\epsilon}(B_1(0))$. The exercise therefore shows that any function in $W^{1,p}(B_1(0))$ that vanishes on $\partial B_1(0)$is bounded.*

4. Assume that $u^j(x) \rightharpoonup u^0(x)$ in $L^2(-\pi,\pi)$. Show that all the Fourier coefficients of u^j converges to the corresponding Fourier coefficients of u^0.

5. * Show that the weak derivatives of the following functions, $f(x)$, either exist or does not exist. Then calculate the weak derivative.

   (a) $f(x) = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ where $f(x)$ is defined on $[-1,1] \subset \mathbb{R}$.

   Does weak derivatives have to be continuous?

   (b) $f(x) = \begin{cases} \frac{1}{x^{1/4}} \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ where $f(x)$ is defined on $[-1,1] \subset \mathbb{R}$.

   (c) $f(x) = \begin{cases} x^{3/4} \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$ where $f(x)$ is defined on $[-1,1] \subset \mathbb{R}$.

   Does weak derivatives have to be bounded?

   (d) $f(x) = \begin{cases} \frac{x_1}{|x|} \text{ if } |x| > 0 \\ 0 \text{ if } x = 0 \end{cases}$ where $f(x)$ is defined on $B_1 \subset \mathbb{R}^3$.

6. * [THE NEED FOR TRACES.] The following exercise is meant to shed light on the trace theorem.

(a) Let $u = \cos(\ln(|x|))$ be a function defined on $(0,1)$. Does $u(x)$ satisfy the assumptions of the trace Theorem? Can you think of any meaningful way to assign a boundary value of $u(x)$ at $x = 0$?[18] and $u = \cos(1/|x|)$ in $\mathbb{R}^n$.

(b) Let $u(x) = \cos(\ln|x|)$ be a function defined on $B_1 \setminus \{0\}$ in $\mathbb{R}^3$. Show that the function $u(x) \in W^{1,2}(B_1 \setminus \{0\})$. However, there is no any meaningful way to ascribe a boundary value of $u(x)$ to the boundary point $x = 0$. Note that the boundary is not the graph of a C^1-function at $x = 0$.

7. ** [A REALLY BAD FUNCTION.] In this exercise we will construct a really bad function - in mathematical analysis we love bad functions as examples.

(a) Show that $u(x) = \begin{cases} \frac{x_1}{|x|^{4/3}} \text{ if } |x| > 0 \\ 0 \text{ if } x = 0 \end{cases}$ satisfies $u(x) \in W^{1,2}(B_2(0))$ when the space dimension $n \geq 3$. Also show that $u(x)$ is not bounded in any neighborhood of $x = 0$.

(b) Since $\mathbb{Q}^3$ is countable we may define a sequence $\{q_j\}_{j=1}^\infty$ such that $\cup_{j=1}^\infty \{q_j\} = \mathbb{Q}^3 \cap B_1^+(0)$. Define $w(x) = \sum_{j=1}^\infty 2^{-j} u(x - q_j)$ and show that $w(x) \in W^{1,2}(B_1^+(0))$ and that $w(x)$ is not bounded, neither from above nor from below, on any open set of $B_1^+(0)$.

[HINT:] *In order to show that $w(x) \in W^{1,2}(B_1^+(0))$ it might be helpful to use the following triangle inequality* $\left\| \sum_j f_j(x) \right\|_{W^{1,2}} \leq \sum_j \|f_j(x)\|_{W^{1,2}}$.

(c) What is $\limsup_{x \to x^0} w(x)$ and $\liminf_{x \to x^0} w(x)$ for $x^0 \in B_1(0) \cap \{x_n = 0\}$?

(d) Does this weird function have well defined boundary values, in the trace sense, on $B_{1/2}(0) \cap \{x_n = 0\}$?

3 The Obstacle Problem.

In this section we will consider *the obstacle problem*. The obstacle problem consists of to minimizing

$$J(u) = \int_{\mathcal{D}} F(\nabla u(x))dx = \int_{\mathcal{D}} |\nabla u(x)|^2 dx, \tag{3.1}$$

in the set

$$K = \{u \in W^{1,2}(\mathcal{D});\ u(x) = f(x) \text{ on } \partial\mathcal{D} \textbf{ and } u(x) \geq g(x) \textbf{ in } \mathcal{D}\}. \tag{3.2}$$

Notice that the set K is the convex set of all functions $u(x)$ that achieves the boundary data $f(x)$ (in the trace sense) and $u(x) \geq g(x)$ in the domain $\mathcal{D}$.

[18] The point of the second question is that you should realize that it is not obvious for a function to have boundary values.

The difference between the obstacle problem and the minimization in Theorem 2.3 is that in the obstacle problem we require that the graph of the minimizer should stay above a prescribed obstacle $g(x)$.

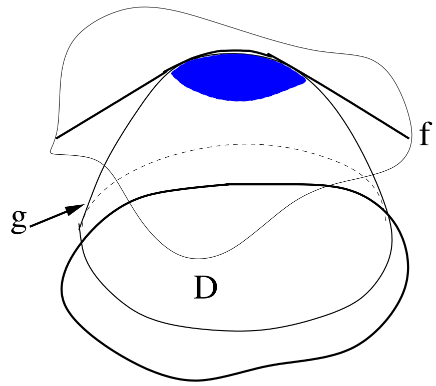


Figure 5: *The graph of a typical solution to the obstacle problem.*

The obstacle problem is much more complicated than the normal Dirichlet problem. For instance, the obstacle problem is non-linear and the obstacle problem has a new unknown: the set where $u(x) = g(x)$ or equivalently the set $\Omega = \{x;\ u(x) > g(x)\}$. Of particular importance is the boundary of the set $\overline{\Omega}$, we call this boundary *The Free Boundary* and denote it $\Gamma = \partial\overline{\Omega}$.

We will denote

$$\Omega = \{x \in \mathcal{D};\ u(x) > g(x)\},\ \Gamma = \partial\overline{\Omega}\ (=\text{Free Boundary}).$$

We are interested in existence of solutions and properties of the solutions and in particular the properties of the sets Ω and Γ.

3.1 Existence of Solutions and some other questions.

To prove that a solution exists is a standard application of Theorem 2.2. In particular we have the following theorem.

Theorem 3.1. *Let $\mathcal{D}$ be a bounded domain with C^1 boundary, $f(x) \in C(\partial\mathcal{D})$ and $g(x) \in W^{1,2}(\mathcal{D})$. Assume furthermore that the set*

$$K = \{u \in W^{1,2}(\mathcal{D});\ u(x) = f(x) \text{ on } \partial\mathcal{D} \textbf{ and } u(x) \geq g(x) \textbf{ in } \mathcal{D}\}.$$

is non-empty. Then there exists a unique function $u(x) \in K$ such that

$$J(u) = \int_{\mathcal{D}} |\nabla u(x)|^2 dx \leq \int_{\mathcal{D}} |\nabla v(x)|^2 dx$$

for all $v(x) \in K$.

Proof: This is proved in the same way as Theorem 2.2. If u^j is a minimizing sequence[19] then $\|\nabla u^j\|_{L^2(\mathcal{D})}$ is bounded and, by Corollary 2.1 $\|u^j\|_{W^{1,2}(\mathcal{D})}$ is

[19]That is a sequence $u^j \in K$ such that $J(u^j) \to \inf_{v\in K} J(v)$.

bounded. We may thus find a sub-sequence $u^k \rightharpoonup u$ in $W^{1,2}(\mathcal{D})$. By Corollary 2.2 $u^0 = f$ on $\partial\mathcal{D}$ and thus $u \in K$. By convexity of the functional $J(u)$ it follows, as in Theorem 2.2, that $J(u) \le \lim_{k\to\infty} J(u^{j_k}) = \inf_{v\in K} J(v)$. □

We have now entirely left the nice and comfortable kind of mathematics where we can explicitly calculate our solutions. Theorem 3.1 is an abstract existence theorem and does not indicate how we should even begin to calculate the minimizer $u(x)$. In general, even for rather nice domains $\mathcal{D}$ and functions $f(x)$ and $g(x)$, we have no idea how to calculate the value of $u(x)$. We have, however, a minimizer $u(x)$ and that minimizer is unique and we would like to describe this minimizer as completely as possible. The questions we will ask are:

1. Does the minimizer $u(x)$ of the obstacle problem satisfy a partial differential equation (as the minimizer to the Dirichlet energy in Theorem 2.3 did.)? We usually call the PDE that the minimizer solves for the Euler-Lagrange equation.
2. Does the minimizer of the obstacle problem satisfy any other "good" properties? Is the minimizer continuous, differentiable or even analytic?
3. What can be said about the set Ω? Is Ω an open set? Is the boundary differentiable? Is there anything that characterize the boundary?[20]

The only thing we know about $u(x)$ is that $u(x) \in K$ and that $u(x)$ is a minimizer of the Dirichlet energy among all functions in K. Therefore, we need to start our investigation with an investigation into what it means to be a minimizer - what variations can we do and what can these variations tell us about the solution.

Variations. If $u(x)$ is a minimizer of $J(u)$ in a convex set K (such as the obstacle problem) and $v \in K$ then, by convexity of K, $(1-t)u + tv \in K$ for all $t \in [0,1]$. Therefore, since $u(x)$ is a minimizer,

$$\int_{\mathcal{D}} |\nabla u(x)|^2 dx \le \int_{\mathcal{D}} |\nabla ((1-t)u + tv)|^2 \, dx \tag{3.3}$$

$$\Rightarrow \int_{\mathcal{D}} \left(2t\nabla u \cdot \nabla(v-u) + t^2|\nabla v|^2\right) \ge 0 \Rightarrow \int_{\mathcal{D}} \nabla u \cdot \nabla(v-u) \ge 0., \tag{3.4}$$

where we divided by $t > 0$ and then sent $t \to 0$ in the last implication.

We can therefore conclude that[21]

$$\text{If } v(x) \in K \text{ then } \int_{\mathcal{D}} \nabla u \cdot \nabla(v-u) \ge 0. \tag{3.5}$$

Furthermore, if $v(x) \in K$ happens to be a function such that $(1-t)u+tv \in K$ for all $t \in (-\epsilon, \epsilon)$ then

$$\int_{\mathcal{D}} \nabla u \cdot \nabla(v-u) = 0. \tag{3.6}$$

[20] In these notes we will not discuss the differentiability properties of Γ.

[21] This condition is usually called *"a variational inequality"*.

This can easily be seen by replicating the argument in (3.3)-(3.4) and using that the inequality reverses direction for $t < 0$.

If we choose the variation $v(x) = u(x)+\phi(x)$, for any $\phi \geq 0$ and $\phi \in W^{1,2}(\mathcal{D})$ with compact support, in (3.5) then we get

$$\int_{\mathcal{D}} \nabla u \cdot \nabla \phi \geq 0. \tag{3.7}$$

And if $\mathrm{spt}(\phi) \subset \{u(x) > g(x)\}$ then we actually get, from (3.6), that

$$\int_{\mathcal{D}} \nabla u(x) \cdot \nabla \phi(x) = 0. \tag{3.8}$$

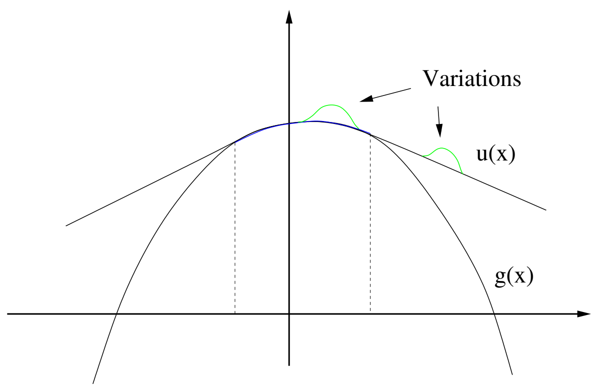


Figure 6: *A graphical representation of the varations, if the bump ϕ is added in a region where $u(x) = g(x)$ (as in the left bump in the graph) then we get the inequality (3.7). But if the bump is added away from the touching set (as the right bump) then we may do variations with $t \in (-\epsilon, \epsilon)$ and thus get the full equality as in (3.8).*

An informal argument and the way ahead: We would like to make an integration by parts in (3.7) in order to deduce that

$$0 \leq \int_{\mathcal{D}} \nabla u \cdot \nabla \phi = \left\{ \begin{array}{l} \text{unjustified int.} \\ \text{by parts} \end{array} \right\} = -\int_{\mathcal{D}} \phi \Delta u, \tag{3.9}$$

Heuristically, since $\phi \geq 0$, the calculation (3.9) implies that $\Delta u \leq 0$ in $\mathcal{D}$. Similarly, from (3.8) we would like to deduce that $\Delta u = 0$ in the set $\{x \in \mathcal{D};\ u(x) > g(x)\}$. Notice that this would directly imply that the equality $u(x) = g(x)$ can only happen whenever $\Delta g(x) \leq 0$. It would thus give us some information about the set Ω (technically about Ω^c).

The problem with the calculation in (3.9) is that is assumes that $u(x)$ has second derivatives. We must first prove that $u(x)$ has second derivatives (in some sense) in order to justify (3.9). This indicates that we need to develop a regularity theory for the obstacle problem.[22] Our next goal will be to show that a solution to the obstacle problem has weak second derivatives. But before we can do that we need to make a simplifying assumption and reformulate the problem slightly.

Normalized solutions to the Obstacle problem: If we assume that $g \in C^2(\mathcal{D})$ and define $v(x) = u(x) - g(x) \geq 0$ then $v(x)$ minimizes

$$\int_{\mathcal{D}} |\nabla(v(x) + g(x))|^2 = \int_{\mathcal{D}} |\nabla v(x)|^2 dx + \int_{\mathcal{D}} 2\nabla v(x) \cdot \nabla g(x) + \int_{\mathcal{D}} |\nabla g(x)|^2 dx =$$

$$= 2\int_{\mathcal{D}} \left(\frac{1}{2}|\nabla v|^2 - v\Delta g(x)\right) dx + \int_{\mathcal{D}} |\nabla g(x)|^2 dx + \int_{\partial\mathcal{D}} v(x)\frac{\partial g(x)}{\partial \nu} dA,$$

where we used an integration by parts in the last equality. Notice that since $g(x)$ is a given function (independent of $v(x)$) the integral $\int_{\mathcal{D}} |\nabla g(x)|^2 dx$ is independent of $v(x)$. Also, since $u \in K$ which means that $u(x) = f(x)$ on $\partial\mathcal{D}$ it follows that $v(x) = u(x) - g(x) = f(x) - g(x)$ on $\partial\mathcal{D}$. In particular,

$$\int_{\mathcal{D}} |\nabla g(x)|^2 dx + \int_{\partial\mathcal{D}} v(x)\frac{\partial g(x)}{\partial \nu} dA =$$

$$= \int_{\mathcal{D}} |\nabla g(x)|^2 dx + \int_{\partial\mathcal{D}} (f(x) - g(x))\frac{\partial g(x)}{\partial \nu} dA$$

is just a constant independent of $v(x)$. It follows that $v(x)$ is a minimizer of the energy

$$\int_{\mathcal{D}} \left(\frac{1}{2}|\nabla v|^2 - v\Delta g(x)\right) dx \tag{3.10}$$

in the set

$$\tilde{K} = \left\{v \in W^{1,2}(\mathcal{D});\ v(x) \geq 0 \text{ and } v(x) = f(x) - g(x) \text{ on } \partial\mathcal{D}\right\}.$$

In many situations it is somewhat easier to work with the function $v(x)$ instead of $u(x)$. It is in particular much easier to work with the formulation with $v(x)$ if $\Delta g(x) = -1$. We therefore make the following definition.

Definition 3.1. *We say that $u(x)$ is a solution to the normalized obstacle problem if $u(x)$ minimizes*

$$\int_{\mathcal{D}} \left(\frac{1}{2}|\nabla u(x)|^2 + u(x)\right) dx \tag{3.11}$$

among all functions in the set

$$K = \left\{u \in W^{1,2}(\mathcal{D});\ u(x) \geq 0 \textit{ and } u(x) = f(x) \textit{ on } \partial\mathcal{D}\right\}.$$

[22] Regularity theory is the part of PDE theory where one proves that solutions to a PDE are more regular, that is have more derivatives, than what is needed to define the solution. Regularity theory also includes deriving a priori estimates of the norm of the solution.

In the rest of these notes we will study solutions to the normalized obstacle problem.

The normalized obstacle problem is somewhat less general than the general obstacle problem. In particular the assumption that $\Delta g(x) = -1$ is a sever limitation. We are however willing to pay the price of a less general problem in order to get a simpler problem.

Exercises:

1. [Comparison principle.] Let $u(x)$ and $v(x)$ be solutions to the normalized obstacle problem in a domain $\mathcal{D}$. Furthermore assume that $v(x) \geq u(x)$ on $\partial\mathcal{D}$. Prove that $v(x) \geq u(x)$ in $\mathcal{D}$.

   Hint: *Assume the contrary, that $u(x) > v(x)$ in some set Σ, and make a variation with $\phi = \max(u(x) - v(x), 0)$.*

2. $*\frac{*}{2}$ Let $u(x)$ and $v(x)$ be as in the previous exercise and assume furthermore that $v(x) > u(x)$ on part of the boundary $\partial\mathcal{D}$ and that $\mathcal{D}$ is connected. Does it follow that $v(x) > u(x)$ in the entire domain $\mathcal{D}$? Would your answer be the same if $u(x)$ and $v(x)$ where harmonic functions?

3. * Let $u(x)$ and $v(x)$ be solutions to the obstacle problem in $\mathcal{D}$ with obstacles $g_u(x)$ and $g_v(x)$ respectively. Assume that $u(x) = v(x)$ on $\partial\mathcal{D}$ and prove that if $g_v(x) \geq g_u(x)$ in $\mathcal{D}$ then $v(x) \geq u(x)$ in $\mathcal{D}$.

4 Regularity Theory.

Let us begin this section, where we will investigate the differentiability properties of solutions to the obstacle problem in $W^{k,2}(\mathcal{D})$, by a simple and classical result.

Proposition 4.1. *Assume that u is a solution to the normalized obstacle problem in $B_R(0)$ for some boundary data. Then, for any $R_0 < R$,*

$$\|\nabla u\|_{L^2(B_{R_0(0)})} \leq \frac{C}{R - R_0} \|u\|_{L^2(B_R(0))}.$$

Proof:'There is only one thing that we know about u; that it is a minimizer to the obstacle problem. 'The only thing we really can do is to compare the energy of u to the energy of an appropriately chosen function.

Let us choose $\phi = \psi^2(x)u(x)$ for some $\psi \in C_c^\infty(B_R(0))$ such that $\psi(x) = 1$ in $B_{R_0}(0)$ and $|\nabla\psi(x)| \leq \frac{2}{|R-R_0|}$. Then, for any $t \in (-1, 1)$,

$$\int_{B_R(0)} \left(\frac{1}{2}|\nabla u|^2 + u\right) dx \leq \int_{B_R(0)} \left(\frac{1}{2}|\nabla(u + t\phi)|^2 + u + t\phi\right) dx$$

which implies that

$$0 \leq \int_{B_R(0)} \left(t\nabla u \cdot \nabla\phi + \frac{t^2}{2}|\nabla\phi|^2 + t\phi\right) dx.$$

If we divide by t and let $t \to 0$ in the last equation we arrive at (also remembering that that t may be either positive or negative)

$$0 = \int_{B_R(0)} (\nabla u \cdot \nabla \phi + \phi)\, dx = \int_{B_R(0)} \left(\psi^2 |\nabla u|^2 + 2u\psi \nabla\psi \cdot \nabla u + \phi\right) dx, \quad (4.1)$$

where we used the definition of ϕ in the last equality.

Rearranging terms in (4.1) we see that

$$\int_{B_R(0)} \psi^2(x)|\nabla u|^2 dx =$$

$$= -\int_{B_R(0)} (2u\psi\nabla\psi \cdot \nabla u + \phi)\, dx \leq -\int_{B_R(0)} 2u\psi\nabla\psi \cdot \nabla u dx \leq \quad (4.2)$$

$$\leq \left(\int_{B_R(0)} \psi^2(x)|\nabla u|^2 dx\right)^{1/2} \left(\int_{B_R(0)} u(x)|\nabla \psi|^2 dx\right)^{1/2},$$

where we used that $\phi = \psi^2 u \geq 0$ in the second line and the Cauchy-Schwartz inequality in the last.

If we divide both sides of (4.2) by $\left(\int_{B_R} \psi^2 |\nabla u|^2 dx\right)^{1/2}$ and then square we get

$$\int_{B_R(0)} \psi^2(x)|\nabla u|^2 \leq \int_{B_R(0)} u(x)|\nabla\psi|^2 dx.$$

The conclusion of the proposition follows if we notice that $\psi = 1$ in $B_{R_0}(0)$ and $|\nabla\psi| \leq \frac{2}{R-R_0}$ and therefore

$$\int_{B_{R_0}(0)} |\nabla u|^2 \leq \int_{B_R(0)} \psi^2(x)|\nabla u|^2 \leq$$

$$\leq \int_{B_R(0)} u(x)|\nabla\psi|^2 dx \leq \frac{4}{(R-R_0)^2} \int_{B_R(0)} u(x) dx.$$

□

4.1 Solutions to the Normalized Obstacle Problem has second derivatives

In this section we will show that a solution to the normalized obstacle problem has weak second derivatives. We begin with a *difference quotient argument*, this is a standard argument in PDE theory and the calculus of variations. In the proof we use the notation $e_1 = (1, 0, 0, .., 0), \cdots, e_i = (0, .., 0, 1, 0, ..)$ for the standard unit vectors.

Lemma 4.1. *Let $u(x)$ be a solution to the normalized obstacle problem in a domain $\mathcal{D}$. Then for each compact set $\mathcal{C} \subset \mathcal{D}$ there exists a constant C only depending on dist$(\mathcal{C}, \mathcal{D}^c)$ such that*

$$\int_{\mathcal{C}} \left| \frac{\nabla(u(x+e_i h) - u(x))}{h} \right|^2 dx \le C \int_{\mathcal{D}} \left| \frac{u(x+e_i h) - u(x)}{h} \right|^2 dx \tag{4.3}$$

for any $h \in \mathbb{R}$ satisfying $|h| < \text{dist}(\mathcal{C}, \mathcal{D}^c)/2$.

Proof: We know that

$$0 \le \int_{\mathcal{D}} (\nabla u(x) \cdot \nabla \phi(x) + \phi(x))\, dx \tag{4.4}$$

for any ϕ with compact support such that $u + t\phi \in K$, that is if $u + t\phi \in K \ge 0$, for $t \in (0, \epsilon)$.

Now we choose $\phi(x) = \psi(x)^2(u(x+e_i h) - u(x))$ for some $\psi \in C_c^\infty(\mathcal{D})$ that satisfies

1. $0 \le \psi(x) \le 1$ for all $x \in \mathcal{D}$,
2. $\psi(x) = 1$ for $x \in \mathcal{C}$,
3. $\psi(x) = 0$ for all x such that $\text{dist}(x, \mathcal{C}) > \frac{\text{dist}(\mathcal{C},\mathcal{D}^c)}{2}$ and
4. $|\nabla \psi(x)| < \frac{4}{\text{dist}(\mathcal{C},\mathcal{D}^c)}$.

Notice that for any $t \in [0, 1)$ we have

$$u(x) + t\phi(x) = t\psi^2(x) u(x+e_i h) + (1 - \psi^2(x)) u(x) \ge 0,$$

since $u(x) \ge 0$. Also $u(x) + t\psi^2(x)(u(x+e_i h) - u(x)) = f(x)$ on $\partial\mathcal{D}$ since $\psi(x) = 0$ on $\partial\mathcal{D}$ and $u(x) = f(x)$ on $\partial\mathcal{D}$.[23]

With this choice of $\phi(x)$ in (4.4) we arrive at

$$\int_{\mathcal{D}} \Big(\nabla u(x) \cdot \nabla(\psi(x)^2(u(x+e_i h) - u(x))) + (\psi(x)^2(u(x+e_i h) - u(x))) \Big) dx \ge 0. \tag{4.5}$$

Next we notice that $u(x + he_i)$ is a minimizer if the normalized obstacle problem in a slightly shifted domain with boundary values $f(x + he_i)$. Arguing similarly as above we arrive at (with $\phi(x) = \psi(x)^2(u(x) - u(x + he_i))$

$$\int_{\mathcal{D}} \Big(\nabla u(x+he_i) \cdot \nabla(\psi(x)^2(u(x) - u(x+he_i))) + (\psi(x)^2(u(x) - u(x+he_i))) \Big) dx \ge 0. \tag{4.6}$$

[23]There is a slight technical detail that should be mentioned here. Since $u(x)$ is defined on $\mathcal{D}$ it follows that $u(x+he_i)$ is defined on the set $\mathcal{D}_{-h} = \{x;\ x+he_i \in \mathcal{D}\} \ne \mathcal{D}$. In particular, the function $u(x) + t\psi(x)^2(u(x+e_i h) - u(x))$ is only defined on $\mathcal{D}_{-h} \cap \mathcal{D}$ which is a strictly smaller set than $\mathcal{D}$. But since $\psi(x) = 0$ on $\mathcal{D} \setminus (\mathcal{D}_{-h} \cap \mathcal{D})$ for $|h| < \text{dist}(\mathcal{C}, \mathcal{D}^c)/2$ we may consider the function that equals $u(x) + t\psi(x)^2(u(x+e_i h) - u(x))$ in $\mathcal{D}_{-h} \cap \mathcal{D}$ and equals zero in $\mathcal{D} \setminus (\mathcal{D}_{-h} \cap \mathcal{D})$ for $|h| < \text{dist}(\mathcal{C}, \mathcal{D}^c)/2$. That function is well defined and all the calculations goes through for that function. It is not uncommon that one uses the simplified convention that an undefined function times zero is zero - it simplifies things.

If we add (4.5) and (4.6) and rearrange the terms we arrive at

$$0 \geq \int_{\mathcal{D}} \left(\nabla(u(x+e_ih)-u(x))\right)\cdot\nabla\left(\psi(x)^2(u(x+e_ih)-u(x))\right)dx =$$

$$= \int_{\mathcal{D}} \left(\psi(x)^2 \left|\nabla(u(x+e_ih)-u(x))\right|^2\right)dx +$$

$$+ \int_{\mathcal{D}} \left(2\psi(x)(u(x+e_ih)-u(x))\nabla\psi(x)\cdot\nabla(u(x+e_ih)-u(x))\right)dx$$

That is

$$\int_{\mathcal{D}} \psi(x)^2 \left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \leq \tag{4.7}$$

$$\leq -\int_{\mathcal{D}} 2\psi(x)(u(x+e_ih)-u(x))\nabla\psi(x)\cdot\nabla(u(x+e_ih)-u(x))dx.$$

In order to continue we use that for any vectors $\mathbf{v},\mathbf{w}\in\mathbb{R}^n$ we have the following inequality $2\mathbf{v}\cdot\mathbf{w}\leq 2|\mathbf{v}|^2+\frac{1}{2}|\mathbf{w}|^2$ which implies that

$$2\psi(x)(u(x+e_ih)-u(x))\nabla\psi(x)\cdot\nabla(u(x+e_ih)-u(x)) =$$

$$= \underbrace{(2(u(x+e_ih)-u(x))\nabla\psi(x))}_{=\mathbf{v}}\cdot\underbrace{(\psi(x)\nabla(u(x+e_ih)-u(x)))}_{=\mathbf{w}} \leq$$

$$\leq 8|\nabla\psi(x)|^2(u(x+e_ih)-u(x))^2 + \frac{1}{2}|\psi(x)|^2\left|\nabla(u(x+e_ih)-u(x))\right|^2 - .$$

Using this in (4.7) we can deduce that

$$\int_{\mathcal{D}} \psi(x)^2 \left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \leq 16\int_{\mathcal{D}} |\nabla\psi(x)|^2(u(x+e_ih)-u(x))^2dx. \tag{4.8}$$

Since $\psi(x)=1$ in $\mathcal{C}$ we can estimate the left side of (4.8) according to

$$\int_{\mathcal{C}} \left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \leq \int_{\mathcal{D}} \psi(x)^2\left|\nabla(u(x+e_ih)-u(x))\right|^2 dx \tag{4.9}$$

and using that $|\nabla\psi|\leq\frac{4}{\text{dist}(\mathcal{C},\mathcal{D}^c)}$ we can estimate the right side of (4.8) according to

$$16\int_{\mathcal{D}} |\nabla\psi(x)|^2(u(x+e_ih)-u(x))^2dx \leq \tag{4.10}$$

$$\leq \frac{256}{\text{dist}(\mathcal{C},\mathcal{D}^c)^2}\int_{\mathcal{D}}(u(x+e_ih)-u(x))^2dx.$$

Putting (4.8), (4.9) and (4.10) and dividing by h^2 we arrive at (4.3). □

Lemma 4.1 provides an integral estimate for the difference quotient of the derivatives. But unless we can also show that the right side in (4.3) is uniformly bounded in h the Lemma would not be very useful. We therefore need the following integral version of the mean value property for the derivatives.

Lemma 4.2. *Assume that $u \in W^{1,2}(\mathcal{D})$ and $\mathcal{C} \subset \mathcal{D}$ is a compact set. Then there exists a constant C depending only on the dimension such that for any $|h| \le dist(\mathcal{C}, \mathcal{D}^c)$*

$$\int_{\mathcal{C}} \left| \frac{u(x+e_i h) - u(x)}{h} \right|^2 dx \le C \int_{\mathcal{D}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx. \tag{4.11}$$

Proof: We will use the following simple version of the Cauchy-Schwartz inequality: *Let $f \in L^2(\Sigma)$ and $|\Sigma|$ denote the area of Σ then*

$$\left| \int_{\Sigma} |f(x)| dx \right|^2 \le |\Sigma| \int_{\Sigma} |f(x)|^2 dx. \tag{4.12}$$

From the fundamental Theorem of calculus[24] we see that

$$\frac{u(x+e_i h) - u(x)}{h} = \frac{1}{h} \int_0^h \frac{\partial u(x+se_i)}{\partial x_i} ds.$$

Thus

$$\int_{\mathcal{C}} \left| \frac{u(x+e_i h) - u(x)}{h} \right|^2 dx = \int_{\mathcal{C}} \left| \frac{1}{h} \int_0^h \frac{\partial u(x+se_i)}{\partial x_i} ds \right|^2 dx \le \tag{4.13}$$

$$\le \int_{\mathcal{C}} \frac{1}{h} \int_0^h \left| \frac{\partial u(x+se_i)}{\partial x_i} \right|^2 dsdx, \tag{4.14}$$

where we used (4.12), with $\Sigma = (0,h)$, in the last inequality. We may continue to estimate (4.14) by using the Fubini Theorem

$$\int_{\mathcal{C}} \frac{1}{h} \int_0^h \left| \frac{\partial u(x+se_i)}{\partial x_i} \right|^2 dsdx = \frac{1}{h} \int_0^h \int_{\mathcal{C}} \left| \frac{\partial u(x+se_i)}{\partial x_i} \right|^2 dxds \le \tag{4.15}$$

$$\le \frac{1}{h} \int_0^h \int_{\mathcal{D}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dxds \le \int_{\mathcal{D}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx. \tag{4.16}$$

Putting (4.13), (4.14), (4.15) and (4.16) together gives the Lemma. □

We continue to prove that boundedness of the integral of the difference quotients implies weak differentiability.

Lemma 4.3. *Let $\mathcal{C} \subset \mathcal{D}$ be a compact set such that $\tilde{\mathcal{C}_\delta} = \{x;\ dist(x,\mathcal{C}) < \delta\} \subset \mathcal{D}$.*

Furthermore assume that $u(x) \in L^2(\mathcal{D})$ and that there exists a constant C such that

$$\int_{\tilde{\mathcal{C}_\delta}} \left| \frac{u(x+e_i h) - u(x)}{h} \right|^2 dx \le C, \tag{4.17}$$

[24] Here again we use that the fundamental theorem of calculus holds for Sobolev functions. This is true in an a.e. sense - but we will simply assume it here.

for all $|h| < \delta$.

Then the weak derivative $\frac{\partial u}{\partial x_i}$ *exists in* $\mathcal{C}$ *and*

$$\int_{\mathcal{C}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx \leq C.$$

Proof: Notice that (4.17) just states that for any sequence $h_j \to 0$ the functions $\frac{u(x+e_i h)-u(x)}{h}$ are bounded in $L^2(\tilde{\mathcal{C}}_\delta)$. Thus, by the weak compactness theorem for L^2-functions, Theorem 2.7, there exists a sub-sequence, still denoted h_j, such that

$$\frac{u(x+e_i h_j)-u(x)}{h_j} \rightharpoonup g_i(x) \in L^2(\tilde{\mathcal{C}}_\delta).$$

By Lemma 2.1 it follows that $\|g_i\|_{L^2(\tilde{\mathcal{C}}_\delta)} \leq C$.

We claim that $g_i(x)$ is the weak x_i-derivative of $u(x)$. To see this we calculate, for any $\phi \in C_c^1(\mathcal{D})$,

$$-\int_{\mathcal{C}} \frac{\partial \phi(x)}{\partial x_i} u(x) dx = \lim_{h_j \to 0} -\int_{\mathcal{C}} \frac{\phi(x+h_j e_i)-\phi(x)}{h_j} u(x) dx =$$

$$= \left\{ \begin{array}{l} \text{Change of var.} \\ x + h_j e_i \to x \\ \text{in } \phi(x+h_j e_i) \end{array} \right\} = \lim_{h_j \to 0} \int_{x - e_i h_j \in \mathcal{C}} \phi(x) \frac{u(x)-u(x-e_i h_j)}{h_j} dx \rightharpoonup$$

$$\rightharpoonup \int_{\mathcal{C}} \phi(x) g_i(x) dx.$$

This proves that $g_i(x) = \frac{\partial u(x)}{dx_i}$. □

We are now ready to formulate the statement of this section as a theorem.

Theorem 4.1. *Let* $u(x)$ *be a solution to the normalized obstacle problem. Then* $u(x)$ *has weak derivatives of second order on any compact subset* $\mathcal{C}$ *of* $\mathcal{D}$ *and there exists a constant* $C_{\mathcal{C}}$ *(depending on* $\mathcal{C}$*) such that for any* i

$$\int_{\mathcal{C}} \left| \nabla \frac{\partial u(x)}{\partial x_i} \right|^2 dx \leq C_{\mathcal{C}} \int_{\mathcal{D}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx, \tag{4.18}$$

in particular

$$\int_{\mathcal{C}} \left| D^2 u(x) \right|^2 dx \leq C_{\mathcal{C}} \int_{\mathcal{D}} \left| \nabla u(x) \right|^2 dx. \tag{4.19}$$

Proof: From (4.3) and (4.11) we see that

$$\int_{\tilde{\mathcal{C}}_\delta} \left| \frac{\nabla(u(x+e_i h)-u(x))}{h} \right|^2 dx \leq C \int_{\mathcal{D}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx, \tag{4.20}$$

where we have used the notation $\tilde{\mathcal{C}}_\delta = \{x;\ \text{dist}(x,\mathcal{C}) < \delta\} \subset \mathcal{D}$ introduced in Lemma 4.3 and chosen $\delta > 0$ small enough so that $\tilde{\mathcal{C}}_\delta \subset \mathcal{D}$.

From Lemma 4.3 and (4.20) we can conclude that $\nabla u(x)$ is weakly differentiable in x_i and

$$\int_{\mathcal{C}} \left| \nabla \frac{\partial u(x)}{\partial x_i} \right|^2 dx \le C_{\mathcal{C}} \int_{\mathcal{D}} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx, \tag{4.21}$$

this proves (4.18).

If we sum (4.21) over $i = 1, 2, ..., n$ the estimate (4.19) and the theorem follows. □

Exercises:

1. * [DIFFERENCE QUOTIENTS AND REGULARITY THEORY.]
   (a) Let $u(x)$ be a minimizer of the Dirichlet energy $\int_{\mathcal{D}} |\nabla u(x)|^2 dx$. Use a difference quotient argument to show that $u \in W^{2,2}(\mathcal{C})$ for any compact set $\mathcal{C} \subset \mathcal{D}$.
   (b) Let $u_i(x) = \frac{\partial u(x)}{\partial x_i}$ and show that for any $\psi \in C_c^2(\mathcal{C})$

   $$\int_{\mathcal{C}} \nabla \phi(x) \cdot \nabla u_i(x) dx = 0.$$

   Conclude that u_i is a minimizer to the Dirichlet energy in $\mathcal{C}$.
   (c) Show, by using induction that, $u \in W^{k,2}(\mathcal{C})$ for any $k \in \mathbb{N}$.
2. * Let $g \in W^{1,2}(\mathcal{D})$ and assume that $u(x)$ minimizes

   $$\int_{\mathcal{D}} \left(|\nabla u(x)|^2 + 2u(x)g(x) \right) dx,$$

   in the set $K = \{u \in W^{1,2}(\mathcal{D});\ u = f \text{ on } \partial\mathcal{D}\}$ where $f(x)$ is some given function. Show that $u \in W^{2,2}(\mathcal{D})$.

   REMARK: *The same is true for $g \in L^2(\mathcal{D})$, can you prove it?***
3. Verify the change of variables in the proof of Lemma 4.3.
4. * Show that the function

   $$u(x) = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$$

   is not a function in $W^{2,2}(-1, 1)$.

4.2 The Euler Lagrange Equations.

Knowing that solutions $u(x)$ to the normalized obstacle problem has second derivatives we are now in position to derive the Euler-Lagrange equations for the obstacle problem. We aim to prove the following theorem.

Theorem 4.2. *Assume $\mathcal{D}$ is a C^1-domain and that $u(x)$ minimizes*

$$\int_{\mathcal{D}} \left(\frac{1}{2}|\nabla u(x)|^2 + u(x) \right) dx \tag{4.22}$$

among all functions in the set

$$K = \left\{ u \in W^{1,2}(\mathcal{D});\ u(x) \geq 0 \text{ and } u(x) = f(x) \text{ on } \partial\mathcal{D} \right\}.$$

Then

$$\begin{array}{ll} \Delta u(x) = \chi_{\{u(x)>0\}} & \text{in } \mathcal{D} \\ u(x) \geq 0 & \text{in } \mathcal{D} \\ u \in W^{2,2}_{loc}(\mathcal{D}), & \end{array}$$

where

$$\chi_{\{u(x)>0\}} = \begin{cases} 1 & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

Proof: That $u(x) \in W^{2,2}_{\text{loc}}(\mathcal{D})$ follows from Theorem 4.1. That $u(x) \geq 0$ follows from the fact that $u \in K$. Therefore we only need to show that

$$\Delta u(x) = \chi_{\{u(x)>0\}}. \tag{4.23}$$

Since $u(x)$ is a minimizer it satisfies the variational inequality

$$\int_{\mathcal{D}} \left(\nabla\phi(x) \cdot \nabla u(x) + \phi(x) \right) dx \geq 0 \tag{4.24}$$

for all $\phi(x) \geq 0$ such that $\phi(x) \in W^{1,2}_0(\mathcal{D})$ where we used the notation

$$W^{1,2}_0(\mathcal{D}) = \{v \in W^{1,2}(\mathcal{D});\ v(x) = 0 \text{ on } \partial\mathcal{D} \text{ in the trace sense.}\}$$

Choosing ϕ with compact support in $\mathcal{D}$ we may, since $u \in W^{2,2}$, integrate by parts in (4.24) and derive

$$0 \leq \int_{\mathcal{D}} \left(-\phi(x)\Delta u(x) + \phi(x) \right) dx = \int_{\mathcal{D}} \phi(x) \left(1 - \Delta u(x) \right) dx.$$

Since $\phi(x)$ is arbitrary this already implies that $0 \leq \Delta u(x) \leq 1$. But we claim something stronger, that $\Delta u(x) = \chi_{\{u(x)>0\}}$. In order to derive this we need to make a more refined choice of $\phi(x)$. To that end we choose $\phi(x) = \psi(x)u(x)$ for some $\psi \in C^1_c(\mathcal{D})$ satisfying $0 \leq \psi(x) \leq 1$. Notice that, with this choice of ϕ it follows that $u(x) + t\phi(x) \in K$ for all $t \in (-1, 1)$. We can conclude that

$$0 = \int_{\mathcal{D}} \phi(x) \left(1 - \Delta u(x) \right) dx = \int_{\mathcal{D}} \psi(x)u(x)(1 - \Delta u(x))dx. \tag{4.25}$$

We will use (4.25) and a contradiction argument to show that $\Delta u(x) = 1$ in the set $\{u(x) \geq \epsilon\}$. Remember that $0 \leq 1 - \Delta u(x) \leq 1$ so if $\Delta u(x) \neq 1$ somewhere in some set $\Sigma \subset \{u(x) \geq \epsilon\}$ then $1 - \Delta u(x) < 0$ in that set. If we choose $\psi(x) \geq 0$ to be a function that is strictly positive in (part of) Σ then (4.25) becomes

$$0 = \int_{\mathcal{D}} \phi(x)\,(1-\Delta u(x))\,dx = \int_{\mathcal{D}} \underbrace{\psi(x)u(x)}_{\geq 0}\underbrace{(1-\Delta u(x))}_{\leq 0}\,dx \leq$$

$$\leq \int_{\Sigma} \psi(x)\,\underbrace{u(x)}_{\geq \epsilon}\underbrace{(1-\Delta u(x))}_{<0}\,dx,$$

this is only possible if Σ has measure zero. We can conclude that $\Delta u(x) = 1$ in $\{u > \epsilon\}$ for any $\epsilon > 0$. Sending $\epsilon \to 0$ we can conclude that $\Delta u(x) = 1$ in the set $\{u(x) > 0\}$.

When $u(x) = 0$ then we naturally have $\Delta u(x) = 0$ at almost every point.[25] Therefore

$$\Delta u(x) = \begin{cases} 1 & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

which is exactly what we wanted to prove. □

Remark: *There is a qurious statement at the end of the proof where we state that $\Delta u(x) = 0$ "at almost every point" of $\{u(x) = 0\}$. The reason for this statement is that, as we will see later, the function*

$$u(x) = \frac{1}{2}(x_n)_+^2 = \begin{cases} \frac{1}{2}x_n^2 & \text{if } x_n > 0 \\ 0 & \text{if } x_n < 0. \end{cases}$$

satisfies $\Delta u(x) = \chi_{\{u(x)>0\}}$. But on the line $x_n = 0$ we have $u(x) = 0$ but $u(x)$ is not even twice differentiable at $x_n = 0$ so $\Delta u(x)$ is not even defined on $\{x_n = 0\}$. Similarly, $u(x) = \frac{1}{2n}|x|^2$ satisfies $\Delta u(x) = 1$ in $\mathbb{R}^n$ so $\Delta u(0) \neq 0$ even though $u(0) = 0$. The almost every means for every x except a set that has zero area. In the above examples the line $\{x_n = 0\}$ and point $\{x = 0\}$ both have zero area and are therefore allowed exceptions. As we already remarked, the theory that we really need in order to make this precise goes beyond this course.

Theorem 4.2 provides us with the Euler-Lagrange equations for solutions to the obstacle problem. In order to continue our investigation of the solutions to the Obstacle problem we would first want to establish that the solutions are continuously differentiable since it is more practical to work with continuously differentiable functions than with the rather abstract space $W^{2,2}(\mathcal{D})$. We would also like to say something about the free boundary $\Gamma = \partial\{u > 0\}$.

[25] This result actually needs to be proved (a proof can be found in pretty much any book on Sobolev spaces). But to prove this we need to understand what *almost every point* means. Since we do not want to use to much measure theory in this course we will accept this final statement on faith. But you must admit - it makes you a little curious to see how this is proved.

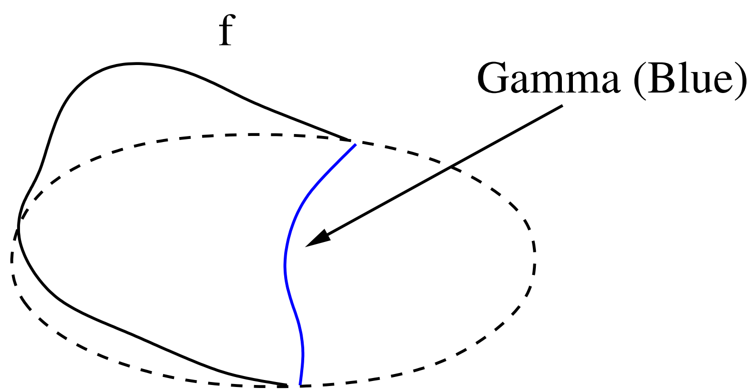


Figure 7: *The graph of a typical solution to the normalized obstacle problem in the ball. The graph takes the values $f(x)$ on the boundary. The solution satisfies $\Delta u(x) = 1$ in part of $\mathcal{D}$ and $u(x) = 0$ in the rest of $\mathcal{D}$. In between we have the free boundary Γ (in blue).*

Exercises:

1. * Show that the (normalized) obstacle problem is non-linear. That is prove that if $u(x)$ and $v(x)$ are solutions to the normalized obstacle problem in $\mathcal{D}$. Then it is, in general, not true that $u(x) + v(x)$ is a solution to the normalized obstacle problem.

2. * Show that for any $\alpha > 0$ the function $u(x) = |x|^{-\alpha}$ belongs to $W^{2,2}(B_1)$ if the space dimension n is large enough. Given an α how large must n be?

5 Continuity of the Solution and its Derivatives.

5.1 Heuristics about the free-boundary.

Theorem 4.2 provides the Euler-Lagrange equations for the normalized obstacle problem. But it does not provide any real information on the free boundary $\Gamma_u = \partial\Omega_u = \partial\overline{\{x;\ u(x) > 0\}}$. In the next section we will show that the solution to the obstacle problem is a continuously differentiable function which implies that both $u(x) = 0$ and $|\nabla u(x)| = 0$ on Γ_u - this is actually rather strong information on the free boundary itself. In this section we will provide some discussion of the free boundary and try to argue that the Euler-Lagrange equations

$$\begin{aligned} &\Delta u(x) = \chi_{\{u(x)>0\}} && \text{in } \mathcal{D} \\ &u(x) \geq 0 && \text{in } \mathcal{D} \\ &u \in W^{2,2}_{\text{loc}}(\mathcal{D}) && \end{aligned} \tag{5.1}$$

actually contains some important information about the free boundary.

In applications the free boundary $\Gamma = \partial\overline{\{v > 0\}}$ is often just as important to understand (and calculate) as the solution itself. In particular, the free boundary often describes the boundary of some set of particular importance - such as the region of ice in a melting problem.

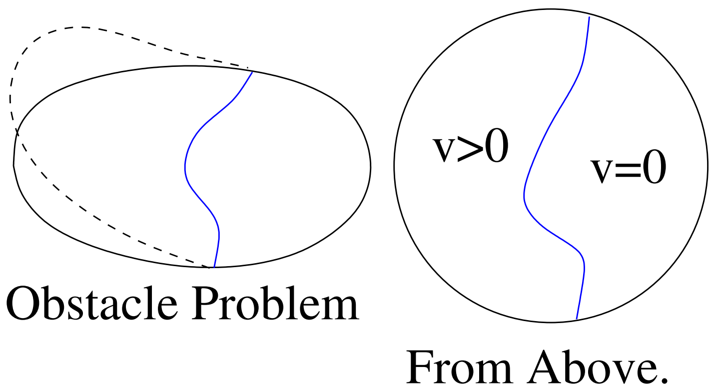


Figure 8: *To the left is the graph of a solution, $v(x)$, to the normalized obstacle problem. In applied problems we are often just as interested in the free boundary Γ, which can be the interface between ice and water in a melting problem. The right picture shows the domain from above with the free boundary marked out. One of the most important questions in free boundary theory is: "Can we describe the free boundary Γ."*

To understand how the free boundary is determined by the Euler-Lagrange equations we we need to understand that The Euler-Lagrange equations (5.1) is not the same as the solution to

$$\begin{array}{ll} \Delta v(x) = 1 & \text{in the set } \{v(x) > 0\} \\ \Delta v(x) = 0 & \text{in the set } \{v(x) = 0\}. \end{array}$$

The information that $u \in W^{2,2}(\mathcal{D})$ provides extra information that specifies the solution.

To see this we consider the one dimensional example.

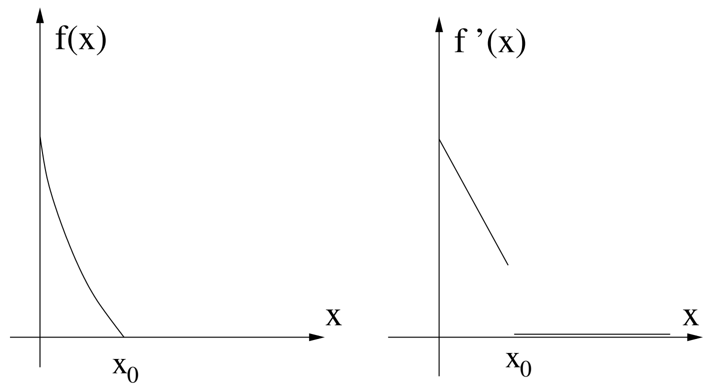
Example: Consider the function

$$f(x) = \begin{cases} \frac{1}{2}(x-1)^2 + (1-x) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } 1 < x < 2. \end{cases}$$

Then

$$\begin{array}{ll} \Delta f(x) = f''(x) = 1 & \text{in the set } \{v(x) > 0\} \\ \Delta f(x) = f''(x) = 0 & \text{in the set } \{v(x) = 0\}, \end{array} \tag{5.2}$$

but $f \notin W^{2,2}([0,2])$.



Then

Figure 9: *The graph of the function $f(x)$ and its derivative $f'(x)$.*

In particular, for small $h > 0$

$$\int_{1/2}^{3/2} \left| \frac{f'(x+h) - f'(x)}{h} \right|^2 \approx \int_{1-h}^{1} \left| \frac{1}{h} \right|^2 dx = \frac{1}{h} \to \infty, \tag{5.3}$$

since $f'(x) = \begin{cases} x-2 & \text{if } x < 1 \\ 0 & \text{if } x > 1. \end{cases}$ It follows that the difference quotients $\frac{f'(x+h)-f'(x)}{h}$ does not converge in L^2. The function $f(x)$ is therefore not a solution to the obstacle problem even though it satisfies the equations (5.2).

From the above example we see that it is not really the following equations that are important:

$$\Delta u(x) = \begin{cases} 1 & \text{in the set } \{u(x) > 0\} \\ 0 & \text{in the set } \{u(x) = 0\}. \end{cases}$$

But the equation

$$\Delta u(x) = \chi_{\{u(x)>0\}},$$

together with the fact that $u \in W^{2,2}(\mathcal{D})$.

The minimization problem *"chooses"* the free boundary $\Gamma = \partial\overline{\{u > 0\}}$ in such way that $u \in W^{2,2}_{\text{loc}}(\mathcal{D})$. From the example above it seems reasonable to conjecture that what determines the position of the free boundary Γ is that the solution should satisfy two boundary conditions, $u(x) = 0$ and $|\nabla u(x)| = 0$ on Γ. The aim of the next section is to prove this.

Exercises:

1. * Find the solution to the following normalized obstacle problem:

$$\text{minimize } \int_{-2}^{2} \left(\frac{1}{2} \left(\frac{\partial u(x)}{\partial x} \right)^2 + u(x) \right) dx$$

in the set

$$K = \left\{ u(x) \in W^{1,2}(-2,2);\ u(-2) = 0,\ u(2) = 2 \text{ and } u(x) \geq 0 \right\}.$$

HINT: *Since this is a one dimensional problem you can calculate it explicitly. Assume that $u > 0$ in $(\gamma, 2]$ and use the equation $\Delta u(x) = 1$ in $(\gamma, 2)$ to calculate $u(x)$. For which γ is $u \geq 0$ satisfies? What is the energy of the function $u(x)$ for a given γ?*

2. Check the calculation (5.3).

5.2 $C^{1,1}-$estimates for the solution.

In this section we aim to prove that the solution satisfies two boundary conditions on the free boundary Γ.

Remember that the Dirichlet problem:

$$\begin{array}{ll} \Delta u(x) = h(x) & \text{in } \mathcal{D} \\ u(x) = f(x) & \text{on } \partial\mathcal{D} \end{array}$$

has a unique solution.[26] This means that the boundary data $f(x)$ and the domain $\mathcal{D}$ determines the value of $\nabla u(x)$ for every $x \in \partial\mathcal{D}$.

For the obstacle problem things are different. The set $\Omega = \{x \in \mathcal{D};\ u(x) > 0\}$ is part of the solution and we might ask what is the criteria that determined the set Ω; or equivalently the free boundary $\Gamma = \partial\overline{\Omega}$. In this section we will prove that the set Ω is the unique set such that:

1. $\Omega \subset \mathcal{D}$,

2. $\text{spt}(f) \subset \overline{\Omega}$ and

3. if $u(x)$ solves the Dirichlet problem

$$\begin{array}{ll} \Delta u(x) = 1 & \text{in } \Omega \\ u(x) = f(x) & \text{on } \partial\Omega \cap \partial\mathcal{D} \\ u(x) = 0 & \text{on } \partial\Omega \setminus \partial\mathcal{D} \end{array}$$

then $|\nabla u| = 0$ on $\partial\Omega \setminus \partial\mathcal{D}$ and $u \geq 0$ in Ω.

This means that Ω is a very special set - and an arbitrarily chosen set $\Sigma \subset \mathcal{D}$ will not satisfy 3.

We begin our proof with a Lemma about solutions to the Poisson equation.

[26] As a matter of fact, we need some mild extra assumption on the solution in order to proclaim uniqueness. For instance, there exists only one solution $u \in C^2(\mathcal{D})$ that is continuous in $\overline{\mathcal{D}}$.

Lemma 5.1. *Let $h(x)$ be a bounded and integrable function, $|h(x)| \leq M$, with support in a bounded set $\mathcal{D} \subset \mathbb{R}^n$, $n \geq 3$, and define*

$$u(x) = -\frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{h(z)}{|x-z|^{n-2}} dz, \tag{5.4}$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$

Then

$$\sup_{x \in \mathbb{R}^n} |u(x)| \leq \frac{M}{2(n-2)} diam(\mathcal{D})^2 \tag{5.5}$$

and there exists constant C_n, that only depend on the dimension, such that

$$|u(x) - u(y)| \leq C_n diam(\mathcal{D}) M |x-y| \tag{5.6}$$

for any $x, y \in \mathbb{R}^n$. Here $diam(\mathcal{D})$ is the diameter of the smallest ball that contains $\mathcal{D}$.

Remark: *Remember that the function $u(x)$ defined as in (5.4) satisfies $\Delta u(x) = h(x)$. A proof of this, for $h(x) \in C_c^2(\mathbb{R}^n)$ can be found in Evans. The result is also true for bounded and integrable $h(x)$. It is also true for more general integrable functions $h(x)$.*

In the Lemma we assume that $n \geq 3$. A similar result is also true for $n = 2$. But when $n = 2$ the Newtonian kernel is logarithmic and thus not bounded at infinity. This leads to some small technical differences in the statement of the theorem. For the sake of brevity we will not include the $\mathbb{R}^2$ case in these notes.

Proof: We begin by proving (5.5). Pick an arbitrary point $x \in \mathbb{R}^n$. Then

$$\begin{aligned} |u(x)| &\leq \frac{1}{(n-2)\omega_n} \left| \int_{\mathbb{R}^n} \frac{h(z)}{|x-z|^{n-2}} dz \right| \leq \\ &\leq \frac{M}{(n-2)\omega_n} \left| \int_{B_{\mathrm{diam}(\mathcal{D})}(x)} \frac{1}{|x-z|^{n-2}} dz \right| \leq \\ &\leq \frac{M}{(n-2)\omega_n} \left| \int_{B_{\mathrm{diam}(\mathcal{D})}(0)} \frac{1}{|z|^{n-2}} dz \right|, \end{aligned} \tag{5.7}$$

where we translated the coordinates $x - z \mapsto z$ in the last step.

If we change to polar coordinates in (5.7) we arrive at

$$|u(x)| \leq \frac{M}{(n-2)} \int_0^{\mathrm{diam}(\mathcal{D})} r dr = \frac{M}{2(n-2)} \mathrm{diam}(\mathcal{D})^2$$

Since x was arbitrary this proves (5.5).

Next we prove (5.6). To that end we pick two points $x, y \in \mathbb{R}^n$. If $|x-y| \geq \mathrm{diam}(\mathcal{D})$ then (5.5) implies that

$$|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq \frac{M}{(n-2)} \mathrm{diam}(\mathcal{D})^2 \leq \frac{M}{(n-2)} \mathrm{diam}(\mathcal{D}) |x-y|.$$

It is therefore enough to prove (5.6) for $|x-y| < \mathrm{diam}(\mathcal{D})$. For the rest of the proof we define $r = |x-y|$ and assume, without loss of generality, that $r < \mathrm{diam}(\mathcal{D})$.

From the defining formula of u we can derive

$$|u(x)-u(y)| = \frac{1}{(n-2)\omega_n}\left|\int_{\mathbb{R}^n}\left(\frac{h(z)}{|x-z|^{n-2}} - \frac{h(z)}{|y-z|^{n-2}}\right)dz\right| \leq$$

$$\leq \frac{1}{(n-2)\omega_n}\left|\int_{\mathbb{R}^n\setminus B_{3r}((x+y)/2)}\left(\frac{h(z)}{|x-z|^{n-2}} - \frac{h(z)}{|y-z|^{n-2}}\right)dz\right| +$$

$$+\frac{1}{(n-2)\omega_n}\left|\int_{B_{3r}((x+y)/2)}\frac{h(z)}{|x-z|^{n-2}}dz\right| +$$

$$+\frac{1}{(n-2)\omega_n}\left|\int_{B_{3r}((x+y)/2)}\frac{h(z)}{|y-z|^{n-2}}dz\right| = I_1 + I_2 + I_3. \tag{5.8}$$

We need to estimate I_1, I_2 and I_3 separately.

We begin to estimate

$$I_2 \leq \frac{M}{(n-2)\omega_n}\int_{B_{3r}((x+y)/2)}\frac{1}{|x-z|^{n-2}}dz \leq$$

$$\leq \frac{M}{(n-2)\omega_n}\int_{B_{4r}(x)}\frac{1}{|x-z|^{n-2}}dz, \tag{5.9}$$

since the integrand is positive and $B_{3r}((x+y)/2) \subset B_{4r}(x)$. Changing to polar coordinates in (5.9) gives

$$I_2 \leq \frac{8M}{(n-2)}r^2.$$

Interchanging the roles of x and y we may estimate I_3 in exactly the same way as we estimated I_2:

$$I_3 \leq \frac{8M}{(n-2)}r^2.$$

It remains to estimate I_1. In order to do that we begin with a simple geometric estimate. By a translation of the coordinate system we may assume that $(x+y)/2 = 0$. If $z \in \mathbb{R}^n \setminus B_{3r}((x+y)/2) = \mathbb{R}^n \setminus B_r(0)$ then, for $t \in [0,1]$,

$$|tx+(1-t)y-z| \geq |z| - |tx+(1-t)y| \geq |z| - r > \frac{|z|}{2}, \tag{5.10}$$

since $|z| \geq 3r$ and $|tx+(1-t)y| = |(2t-1)x| < r$ if $x+y=0$ and $|x-y| = r$.

Using the fundamental theorem of calculus we can also estimate

$$\left|\frac{1}{|x-z|^{n-2}} - \frac{1}{|y-z|^{n-2}}\right| = \left|\int_0^1 \frac{d}{dt}\frac{1}{|tx+(1-t)y-z|^{n-2}}dt\right| =$$

$$= (n-2)\left|\int_0^1 \frac{(x-y)\cdot(tx+(1-t)y-z)}{|tx+(1-t)y-z|^n}dt\right| \leq \tag{5.11}$$

$$\leq (n-2)|x-y|\int_0^1 \frac{1}{|tx+(1-t)y-z|^{n-1}}dt \leq$$

$$\leq \frac{2^{n-1}(n-2)|x-y|}{|z|^{n-1}},$$

where we used (5.10) in the last inequality.

Using (5.11) we may estimate

$$I_1 = \frac{1}{(n-2)\omega_n}\left|\int_{\mathbb{R}^n\setminus B_{3r}(0)}\left(\frac{h(z)}{|x-z|^{n-2}} - \frac{h(z)}{|y-z|^{n-2}}\right)dz\right| \leq$$

$$\leq \frac{M}{(n-2)\omega_n}\left|\int_{\mathrm{spt}(h)\setminus B_{3r}(0)}\left(\frac{1}{|x-z|^{n-2}} - \frac{1}{|y-z|^{n-2}}\right)dz\right| \leq$$

$$\leq \frac{2^{n-1}M|x-y|}{\omega_n}\int_{\mathrm{spt}(h)\setminus B_{3r}(0)}\frac{1}{|z|^{n-1}}dz. \tag{5.12}$$

The last integral in (5.12) can be estimated by noticing that[27]

$$\int_{\mathrm{spt}(h)\setminus B_{3r}(0)}\frac{1}{|z|^{n-1}}dz \leq \int_{B_{\mathrm{diam}(\mathcal{D})}}\frac{1}{|z|^{n-1}}dz =$$

$$= \omega_n\int_0^{\mathrm{diam}(\mathcal{D})} ds = \omega_n \mathrm{diam}(\mathcal{D}),$$

where we changed to polar coordinates in the final step. Using this in (5.12) we can conclude that

$$I_1 \leq 2^{n-1}M|x-y|\mathrm{diam}(\mathcal{D}).$$

Inserting the estimates of I_1, I_2 and I_3 in (5.8) we can conclude that

$$|u(x)-u(y)| \leq \frac{16M}{(n-2)}|x-y|^2 + 2^{n-1}M|x-y|\mathrm{diam}(\mathcal{D}) \leq$$

$$\leq \left(\frac{16}{(n-2)} + 2^{n-1}\right)\mathrm{diam}(\mathcal{D})M|x-y|. \tag{5.13}$$

Noticing that the quantity in the brackets in (5.13) only depend on the dimension we may call that quantity c_n. This proves (5.6) $\square$

[27]The integral increases if $\mathrm{spt}(h)$ is centered at the origin where $1/|z|^{n-1}$ is large. The integral therefore achieves its maximum if $\mathrm{spt}(h) = B_{\mathrm{diam}(\mathcal{D})(0)}$.

Lemma 5.2. *Assume that $h(x)$ is a bounded, $|h(x)| \le M$, and integrable function in $B_3(0)$ and that*

$$\begin{array}{ll} \Delta u = h(x) & \text{in } B_3(0) \\ \sup_{B_3(0)} u(x) \le N. & \end{array}$$

Then

$$|u(x) - u(y)| \le C_n\,(M+N)\,|x-y| \qquad \text{for all } x, y \in B_2(0).$$

Proof: If we define

$$v(x) = -\frac{1}{(n-2)\omega_n}\int_{B_3(0)} \frac{h(z)}{|x-z|^{n-2}}dz,$$

then, by Lemma 5.1,

$$|v(x) - v(y)| \le C_n M|x-y|. \tag{5.14}$$

And, since $v(x)$ is defined by means of a convolution by the Newtonian kernel, $\Delta v(x) = h(x)$.

Also the function $w(x) = u(x) - u(y)$ satisfies

$$\Delta w(x) = \Delta u(x) - \Delta v(x) = h(x) - h(x) = 0$$

and

$$\sup_{B_3(x)} |w(x)| \le \sup_{B_3(x)} |u(x)| + \sup_{B_3(x)} |v(x)| \le N + \frac{9M}{2(n-2)},$$

where we used Lemma (5.1) in the last inequality.

Since $w(x)$ is harmonic we may use the following estimate[28] of the derivatives of $w(x)$

$$|\nabla w(x)| \le C\|w\|_{L^1(B_1(x))} \le C_n(N+M) \qquad \text{for any } x \in B_2(0). \tag{5.15}$$

From (5.15) and the mean value property for the derivative we can conclude that for any $x, y \in B_2(0)$ there exists a ξ between x and y such that.

$$|w(x) - w(y)| \le |\nabla w(\xi)||x-y| \le C_n(N+M)|x-y| \tag{5.16}$$

Finally we may use (5.14), (5.16) and the triangle inequality to conclude that for any $x, y \in B_2(0)$

$$|u(x) - u(y)| = |(w(x) - w(y)) + (v(x) - v(y))| \le$$

$$\le |w(x) - w(y)| + |v(x) - v(y)| \le C_n\,(M+N)\,|x-y|,$$

where C_n may be different from the constant C_n in (5.16). □

[28] See for instance Theorem 7 in section 2.2 in Evans

Corollary 5.1. *Let $h^k(x)$ be a sequence of uniformly bounded, $|h^k(x)| \leq M$, and integrable functions and $u^k(x)$ be a sequence of functions that satisfies*

$$\begin{array}{ll} \Delta u^k = h^k(x) & \text{in } B_3(0) \\ \sup_{B_3(0)} |u^k(x)| \leq N. & \end{array}$$

Then there exists a function u^0 and a subsequence u^{k_j} such that $u^{k_j} \to u^0$ uniformly in $B_2(0)$

Proof: The sequence u^k is equicontinuous by Lemma 5.2. By the Arzela-Ascoli Theorem we may extract a uniformly converging sub-sequence. □

Corollary 5.2. *Let $u(x)$ be a solution to the normalized obstacle problem in $\mathcal{D}$. Then the set $\Omega = \{x \in \mathcal{D};\ u(x) > 0\}$ is open.*

Proof: This follows from the continuity of the solution to $\Delta u(x) = \chi_{\{u>0\}}$. □

Lemma 5.3. [Comparison Principle.] *Let $f(x)$ and $g(x)$ be bounded functions in a bounded domain $\mathcal{D}$. Furthermore assume that $\Delta u(x) = f(x)$ and $\Delta v(x) = g(x)$ in $\mathcal{D}$.*[29] *Then if $f(x) \geq g(x)$ in $\mathcal{D}$ and $u(x) \leq v(x)$ on $\partial\mathcal{D}$ it follows that*

$$u(x) \leq v(x) \qquad \text{in } \mathcal{D}$$

Proof:[30] It is enough to show that $w(x) = u(x) - v(x) \leq 0$ in $\mathcal{D}$. That is we need to show that any function $w(x)$ that satisfies

$$\begin{array}{ll} \Delta w(x) = f(x) - g(x) \geq 0 & \text{in } \mathcal{D} \\ w(x) = u(x) - v(x) \leq 0 & \text{on } \partial\mathcal{D} \end{array} \tag{5.17}$$

will be non-positive.

Notice that w is the minimizer of

$$\int_{\mathcal{D}} \left(\frac{1}{2} |\nabla w(x)|^2 + w(x)(f(x) - g(x)) \right) dx \tag{5.18}$$

in $K = \{w \in W^{1,2}(\mathcal{D});\ w = u(x) - v(x) \text{ on } \partial\mathcal{D}\}$ since the Euler-Lagrange equations of (5.18) is $\Delta w(x) = f(x) - g(x)$ and the solution is unique.

By the function $\tilde{w}(x) = \min(w(x), 0) \in K$ and clearly

$$\int_{\mathcal{D}} \left(\frac{1}{2} |\nabla \tilde{w}(x)|^2 + \tilde{w}(x)(f(x) - g(x)) \right) dx = \tag{5.19}$$

$$= \int_{\mathcal{D}\cap\{w\leq 0\}} \left(\frac{1}{2} |\nabla \tilde{w}(x)|^2 + \tilde{w}(x)(f(x) - g(x)) \right) dx = \tag{5.20}$$

$$= \int_{\mathcal{D}\cap\{w\leq 0\}} \left(\frac{1}{2} |\nabla w(x)|^2 + w(x)(f(x) - g(x)) \right) dx \leq \tag{5.21}$$

[29] Assume for instance that $u, v \in W^{2,2}(\mathcal{D})$ in order to make sense of these equations.

[30] The proof is more or less the same as in a previous exercise.

$$\leq \int_{\mathcal{D}} \left(\frac{1}{2}|\nabla w(x)|^2 + w(x)(f(x) - g(x)) \right) dx, \tag{5.22}$$

with equality only if $w(x) \leq 0$ in $\mathcal{D}$. But since w is a minimizer we must have equality in (5.19)-(5.22). This finishes the proof. □

Theorem 5.1. *Let $u(x)$ be a solution to the normalized obstacle problem in the domain $\mathcal{D}$. Assume furthermore that for some $s > 0$*

$$x^0 \in \Gamma \cap \{x \in \mathcal{D};\ dist(x, \partial\mathcal{D}) > s\}.$$

Then there exists a constant C, depending on the dimension n, such that

$$\sup_{x \in B_r(x^0)} u(x) \leq Cr^2 \qquad \textit{for every } r \leq \frac{s}{2},$$

where the constant C depend only on the dimension.

Remark: *Notice that the constant C does not depend on the solution.*
Proof: We will prove the Theorem in several steps.

Step 1: *Reduction to the statement that it is enough to prove that: If $u(x)$ is a solution to the normalized obstacle problem in $B_2(0)$ such that $u(0) = 0$ then $\sup_{B_1(0)} u \leq C$ for some C depending only on the dimension.*

Proof of step 1: Let $u(x)$ and x^0 is as in the Theorem. Then the function

$$u_r(x) = \frac{u(rx + x^0)}{r^2}$$

will satisfy $u_r \in W^{2,2}(B_2(0))$ and

$$\Delta u_r(x) = \Delta \left(\frac{u(rx + x^0)}{r^2} \right) = \Delta u(y)\lfloor_{y = rx + x^0} =$$

$$= \left\{ \begin{array}{l} 1 \text{ if } u(rx + x^0) > 0 \Rightarrow u_r(x) > 0 \\ 0 \text{ if } u(rx + x^0) = 0 \Rightarrow u_r(x) = 0, \end{array} \right\} = \chi_{\{u_r > 0\}}$$

in the set $x \in \{x;\ rx + x^0 \in \mathcal{D}\}$. This is a convoluted way of trying to indicate how the chain rule implies that

$$\Delta u_r(x) = \chi_{\{u_r(x) > 0\}} \qquad \text{in } \{x;\ rx + x^0 \in \mathcal{D}\}.$$

Notice that if $B_s(x^0) \subset \mathcal{D}$ then $B_2(0) \subset \{x;\ rx + x^0 \in \mathcal{D}\}$. Thus u_r solves the normalized obstacle problem in $B_2(0)$ and $u_r(0)$.

Thus if any solution $v(x)$ to the normalized obstacle problem in $B_2(0)$ that satisfies $v(0) = 0$ satisfies $\sup_{B_1(0)} v(x) \leq C$ then this applies to u_r. We may conclude that

$$\frac{u(rx + x^0)}{r^2} = u_r(x) \leq C \qquad \text{for every } x \in B_1(0).$$

But this implies that

$$\sup_{x\in B_r(x^0)} u(x) \leq C.$$

Step 1 is therefore proved.

Step 2: *If $u(x)$ is a solution to the normalized obstacle problem in $B_2(0)$ and $u(0)=0$ then ther exists a constant c_n such that if $y\in \partial B_1(0)$ then*

$$u(x)\geq c_n u(y)-\frac{1}{2n} \qquad \text{for all } x\in B_{1/2}(0)\cap \partial B_1(0).$$

Proof of Step 2: Since $\Delta u(x)\leq 1$ it follows from the comparison principle that $u(x)\leq v(x)$ where $v(x)$ is defined by

$$\begin{aligned} \Delta v(x) &= 0 && \text{in } B_1(y)\\ v(x) &= u(x) && \text{on } \partial B_1(y). \end{aligned} \tag{5.23}$$

If we define $w(x)=v(x)-\frac{1}{2n}+\frac{1}{2n}|x-y|^2$ then

$$\begin{aligned} \Delta w(x) = 1 \geq \Delta u(x) && \text{in } B_1(y)\\ w(x) = u(x) && \text{on } \partial B_1(y). \end{aligned}$$

We may conclude that

$$v(x)-\frac{1}{2n}\leq w(x)\leq u(x)\leq v(x) \qquad \text{in } B_1(y). \tag{5.24}$$

Since $v(x)\geq u(x)\geq 0$ and $v(y)\geq u(y)$ we may conclude from the Harnack inequality that, for some constant C_n only depending on the dimension,

$$v(y)\leq \sup_{B_{1/2}} v(x)\leq C_n \inf_{B_{1/2}(y)} v(x) \Rightarrow \inf_{B_{1/2}(y)} v(x)\geq \frac{v(y)}{C_n}\geq \frac{u(y)}{C_n}, \tag{5.25}$$

where we also used that $v(y)\geq u(y)$ in the last inequality.

But from (5.24) and (5.25) we can conclude that

$$\frac{u(y)}{C_n}\leq \inf_{B_{1/2}(y)} v(x)\leq \inf_{B_{1/2}(y)} u(x)+\frac{1}{2n} \Rightarrow \frac{u(y)}{C_n}-\frac{1}{2n}\leq \inf_{B_{1/2}(y)} u(x).$$

This proves step 2 with $c_n=\frac{1}{C_n}$.

Step 3: *Assume that $u(x)$ is a solution to a normalized obstacle problem in $B_2(0)$ such that $u(0)=0$ then $\sup_{B_1(0)} u(x)\leq C_n$ for some universal constant C_n depending only on the dimension.*

Proof of Step 3. If we let $v(x)$ be the function defined by

$$\begin{aligned} \Delta v(x) &= 0 && \text{in } B_1(0)\\ v(x) &= u(x) && \text{on } \partial B_1(0). \end{aligned} \tag{5.26}$$

Then we may argue as in step 2 to conclude that that $v(x) - \frac{1}{2n} \le u(x) \le v(x)$. Since $u(0) = 0$ we can conclude that

$$v(0) \le \frac{1}{2n}. \tag{5.27}$$

By the mean-value property for harmonic functions we can conclude from (5.27) that

$$\frac{1}{2n} \ge \frac{1}{\omega_n} \int_{\partial B_1(0)} v(x)dx = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x)dx, \tag{5.28}$$

since $v(x) = u(x)$ on $\partial B_1(0)$.

If $u(y) = \sup_{x \in \partial B_1(0)} u(x)$ then we may estimate the right side in (5.28) according to

$$\frac{1}{2n} \ge \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x)dx \ge \frac{1}{\omega_n} \int_{\partial B_1(0) \cap B_{1/2}(y)} u(x)dx \ge \tag{5.29}$$

$$\ge \frac{1}{\omega_n} \int_{\partial B_1(0) \cap B_{1/2}(y)} \inf_{z \in B_{1/2}(y)} u(z)dx \ge \frac{K}{\omega_n} \left(c_n u(y) - \frac{1}{2n} \right),$$

where $K = \int_{\partial B_1(0) \cap B_{1/2}(y)} dA$ and we used that $u \ge 0$ in the second inequality and Step 2 as well as the fact that $B_{1/2}(y) \cap \partial B_1(0)$ consists of a fixed proportion of $\partial B_1(0)$ in the last inequality.

Rearranging the terms in (5.29) we arrive at

$$u(y) \le \frac{K+1}{2Kc_n},$$

where the right side depend only on the dimension. This finishes the proof. □

Corollary 5.3. *If u is a solution to the normalized obstacle problem in a domain $\mathcal{D}$. Then $|\nabla u(x)| = 0$ for any point $x \in \Gamma$.*

Proof: Let $x^0 \in \Gamma \cap \mathcal{D}$. We need to show that

$$\lim_{x \to x^0} \frac{u(x) - u(x^0)}{|x - x^0|} = 0.$$

But if we use the notation $r = r(x) = |x - x^0|$ then it directly follows from Theorem 5.1 and the assumption $x^0 \in \Gamma$ (which implies that $u(x^0) = 0$ since u is continuous by Lemma 5.2) that

$$\left| \frac{u(x) - u(x^0)}{|x - x^0|} \right| \le \frac{Cr^2}{r} = Cr \to 0 \text{ as } r \to 0.$$

The Corollary follows. □

Theorem 5.2. *Let $u(x)$ be a solution to the normalized obstacle problem in a domain $\mathcal{D}$. Then there exists a constant C_n depending only on the dimension such that if $u(y) = 0$ and $B_{s/4}(y) \subset \mathcal{D}$ then*

$$|D^2u(x)| \leq C_n \qquad \text{for every } x \in B_{s/8}(y) \cap \{u > 0\}.$$

Furthermore, $u(x)$ is analytic in $\Omega = \{u(x) > 0\}$.

Proof: Let $y \in \mathcal{D}$ be any point such that $u(y) = 0$. Also let $s = \text{dist}(y, \partial\mathcal{D})$ so that $B_s(y) \subset \mathcal{D}$ and $z \in B_{s/8}(y) \cap \{u > 0\}$.

Next we consider the largest ball $B_r(z) \subset \{u > 0\}$ and pick any point $q \in \partial B_r(z) \cap \partial\{u > 0\}$. Notice that since $z \in B_{s/8}(y)$ and $u(y) = 0$ it follows that $r \leq s/8$.

We also claim that $B_{4r}(q) \subset \mathcal{D}$. By the triangle inequality $|y - q| \leq |y - z| + |z - q| < s/8 + r < s/2$ which implies that $B_{4r}(q) \subset B_{s/2}(q) \subset B_s(y) \subset \mathcal{D}$.

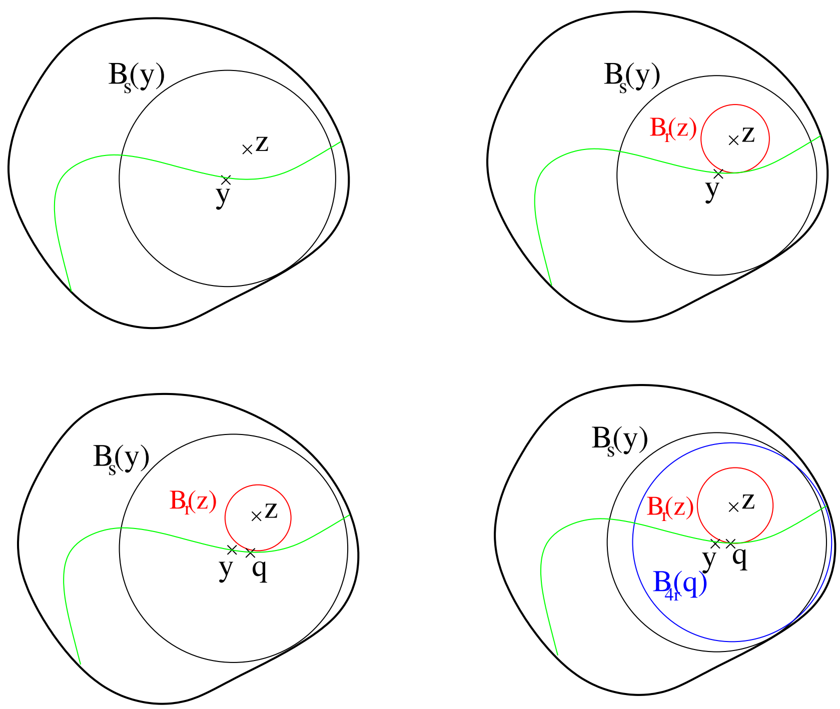


Figure 10: *The above figure (not drawn to scale) tries to indicate how $B_r(z)$ and $B_{4r}(q)$ is choosen. We have the point y such that $B_s(y) \subset \mathcal{D}$ shown in the first figure. The ball $B_r(z)$ (in red) is then choosen to be the largest ball contained in $\Omega = \{u > 0\}$. That means that $\partial B_r(z)$ touches the free boundary (the green curve) in some point q as shown in the third picture. We then choose the ball $B_{4r}(q)$ (in blue) as in the last picture. The purpose of this construction*

is that since $B_{4r}(q) \subset \mathcal{D}$ we know, Theorem 5.1, that $u(x)$ is uniformly bounded in $B_r(z) \subset B_{4r}(q)$.

By Theorem 5.1 it follows that

$$\sup_{x\in B_{2r}(q)} u(x) \leq 4Cr^2.$$

This in particular implies that

$$\sup_{x\in B_r(z)} u(x) \leq \sup_{x\in B_{2r}(q)} u(x) \leq 4Cr^2$$

since $B_r(z) \subset B_{2r}(q)$.

This implies that

$$\left.\begin{array}{l} \Delta u(x) = 1 \text{ in } B_r(z) \\ u(x) \leq 4Cr^2 \text{ in } B_r(z) \end{array}\right\} \Rightarrow \left\{ \begin{array}{ll} \Delta\left(u(x) - \frac{1}{2n}|x-z|^2\right) = 0 & \text{in } B_r(z) \\ \left|u(x) - \frac{1}{2n}|x-z|^2\right| \leq \left(4C + \frac{1}{2n}\right) r^2 & \text{in } B_r(z). \end{array}\right.$$

Using standard estimates on derivatives for harmonic functions[31] we can conclude that, at the point $x = z$,

$$\left|D^2 u(x) - \frac{1}{2n}|x-z|^2\right| \leq C_2\left(4C + \frac{1}{2n}\right).$$

But this clearly implies that

$$|D^2 u(z)| \leq C_n,$$

where C_n is a constant that only depend on the dimension.

That u is analytic follows from the fact that $u - \frac{1}{2n}|x-z|^2$ is harmonic in a small neighborhood around z and that harmonic functions are analytic. □

Exercises:

1. * Let $u(x)$ be a minimizer of the normalized obstacle problem in $B_1(0) \subset \mathbb{R}^3$ with constant boundary values $u(x) = t$ on $\partial B_1(0)$. Calculate $u(x)$ and show that the free boundary Γ is given by a sphere of radius s. Determine the relation between t and s.

   HINT: *If we write $u(x) = u(r)$ where $r = |x|$ then we get a one dimensional problem with the following conditions $u(1) = t$, $u(s) = |\nabla u(s)| = 0$. Since $u(r) - \frac{1}{6}r^2$ is harmonic in $\{u(r) > 0\}$ we should be able to write $u(r) - \frac{1}{6}r^2 = \frac{c}{r} + d$ for two constants c and d. Since also s is unknown we have three unknown and three boundary conditions to satisfy.*

2. Verify all the calculations in the proof of Lemma 5.1.

[31] $|D^2 h(z)| \leq \frac{C_2}{r^{n+2}}\|h\|_{L^1(B_r(z))} \leq \frac{C_2}{r^2}\|h\|_{L^\infty(B_r(z))}$ see Evans Theorem 7 chapter 2.2.

3. ** Let $u(x)$ be a solution to the normalized obstacle problem in a connected domain $\mathcal{D}$. Show that for any set $\mathcal{C}$ such that $\text{dist}(\mathcal{C}, \partial\mathcal{D}) \geq \delta > 0$ there exists a constant C such that if $u(x) \geq C$ for any point $x \in \mathcal{C}$ then $\Gamma \cap \mathcal{C} = \emptyset$.

4. ** Let $u(x)$ be a solution to the normalized obstacle problem in $\mathcal{D}$. Show that if $x^0 \in \Gamma$ then $\sup_{x \in B_r(x^0)} u(x) \geq \frac{r^2}{2n}$.

   HINT: *Let y be a point arbitrarily close to x^0 such that $u(y) > 0$. Argue by contradiction and assume that $u(y) - \frac{1}{2n}|x-y|^2$ is strictly negative on $\partial B_r(y)$. What equation does $u(y) - \frac{1}{2n}|x-y|^2$ solve in $\Omega \cap B_r(y)$? What are the boundary values of $u(y) - \frac{1}{2n}|x-y|^2$ on $\partial(\Omega \cap B_r(y))$?*

6 The Free Boundary - some background and motivation.

So far we have shown that if $u(x)$ is a solution to the normalized obstacle problem in $\mathcal{D}$ then $u(x)$ is a continuously differentiable function that is analytic in $\Omega = \{u > 0\}$. Can we say anything else regarding the free boundary $\Gamma = \partial\{u > 0\}$? Ideally we would like the free boundary to be a smooth, say C^∞ or even analytic, surface in $\mathcal{D}$. In this section we will sketch some theory regarding the free boundary. However, we will not provide many proofs since the material goes beyond this course. We begin with two examples that shows that the free boundary is not even C^1.

Counterexamples to $\Gamma \in C^1$. The free boundary isn't C^1- in general. The following to geometries can be shown to exist.

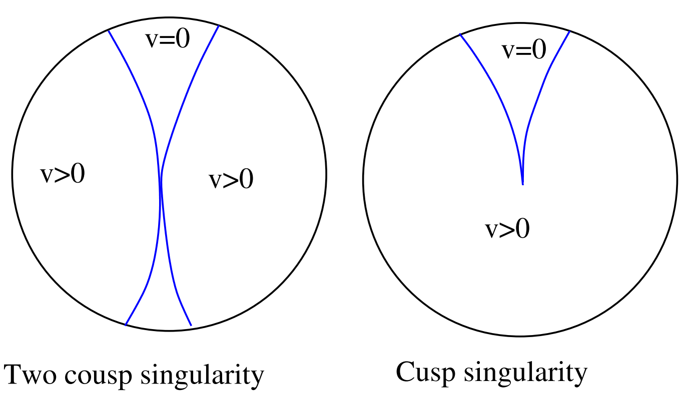


Figure : *The above two geometries can be shown to exist for free boundaries. In the graph we depict the domain where a solution $v(x)$ to the normalized obstacle problem is defined. The free boundary is marked with a blue line.*

But we might hope to prove that Γ is C^1 around almost every point. Here we use almost every in the technical sense that means that there is a set of zero surface area that contains all the singularities of the free boundary. It is indeed the case that the free boundary is C^1 around every point in an open set whose complement has area zero.

In order to prove this we need two technical theorems that we state here without proof.

Theorem 6.1. [CAFFARELLI] *Let $v(x)$ be a solution to the obstacle problem in $\mathcal{D}$ and $\Omega = \{v(x) > 0\}$. Then Ω has locally finite perimeter.*[32]

Notice that this theorem excludes very many bad possibilities such as Ω being a Koch snow-flake.

The second theorem we need is a deep result by E. de Giorgi.

Theorem 6.2. [DE GIORGI] *If Ω has finite perimeter then Ω has a measure theoretic normal at a.e. point of its boundary.*

In order to understand the theorem of de Giorgi we need to understand what a measure theoretic normal is. This requires more measure theory than we assume in this course. But basically, the measure theoretic normal is a unit vector valued function living on the boundary of a set that agrees with the usual normal at every point where the boundary is continuously differentiable. If we call this vector valued function $\nu(x)$ the de Giorgi theorem says that at almost every point $z \in \partial\Omega$ of the boundary of a set Ω of finite perimeter

$$\lim_{r \to 0} \frac{\int_{\partial\Omega \cap B_r(z)} \nu(x)}{\int_{\partial\Omega \cap B_r(z)} |\nu(x)|} = \nu(z), \tag{6.1}$$

where $\nu(z)$ has unit length.

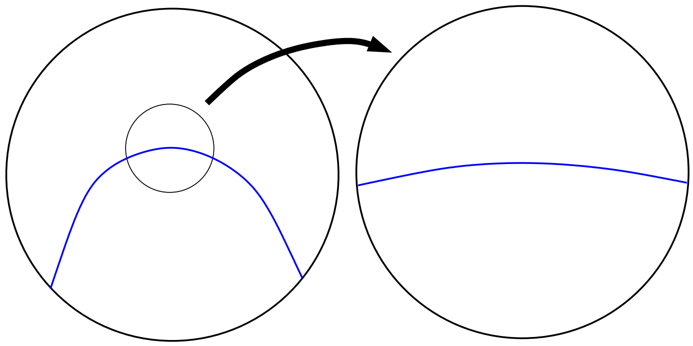
Equation (6.1) states that the measure theoretic normal points in the same direction at almost every point of the boundary. It is also clear from (6.1) that we would need to develop a theory for integration of bad sets in order to make sense of the integrals.

However, it is not difficult to believe that if the normal points in almost the same direction in small balls $B_r(z)$ centered at z then the boundary is close to a hyperplane in small balls around z. In particular, if all the normals point in the same direction then the boundary of the set must be a hyperplane.

If Ω has a measure theoretic normal at $z \in \partial\Omega$ then there exists a unit vector $\nu(z)$ such that

$$\Omega_r(z) = \{x;\ rx + z \in \Omega\} \to \{x;\ x \cdot \nu(z) < 0\} \text{ as } r \to 0.$$

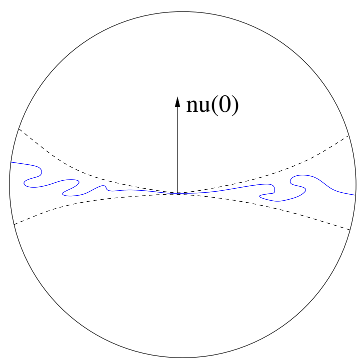
[32] A set of finite perimeter is just a set of finite area. However, "area" must be defined in a very special sense since a set may have finite perimeter without being differentiable.



Rescaling a small ball into a unitball will increase the flatness.

Figure : *When we rescale the set Ω to Ω_r the ball $B_r(z)$ is stretched out to become a unit ball. In the process the boundary becomes flatter.*

There are however really bad sets that has measure theoretic normals as the next picture shows. As a matter of fact pretty much any set we can imagine, without years of mathematical training, will be a set with well defined measure theoretic normal at almost every point.



Bad set with Measure Theo. Normal.

Figure : *A set where there is a well defined normal at the origin even though it is not a very regular set.*

We may however use the de Giorgi and Caffarelli Theorems to gain some information regarding the solutions to the normalized obstacle problem.

Lemma 6.1. *Let $u(x)$ be a solution to the normalized obstacle problem in a domain $\mathcal{D}$. Assume furthermore that $z \in \Gamma_u$ and that for some unit vector $\nu(z)$*

$$\Omega_r(z) = \{x;\ rx + z \in \Omega\} \to \{x;\ x \cdot \nu(z) < 0\} \text{ as } r \to 0.^{33} \tag{6.2}$$

[33] We know from the de Giorgi and Caffarelli Theorems that this is satisfied at almost every point.

Then

$$u_r(x) = \frac{u(rx+z)}{r^2} \to u_0 = \frac{(\nu(z)\cdot x)_-^2}{2} \tag{6.3}$$

locally uniformly in $\mathbb{R}^n$*. Here* $(\nu(z)\cdot x)_- = \max(-\nu(z)\cdot x, 0)$*.*

Sketch of the Proof: By Theorem 5.1 the functions $u_r(x)$ are locally uniformly bounded. Furthermore

$$\Delta u_r(x) = \Delta \frac{u(rx+z)}{r^2} = \chi_{\{u(rx+z)>0\}} = \chi_{\{u_r(x)>0\}} \to \chi_{\{x;\, x\cdot\nu(z)<0\}}, \tag{6.4}$$

where we used (6.2) in the limit to the right.

Furthermore, by the bounds in Theorem 5.2, $u_r \to u_0$ in $C^{1,\alpha}$ locally.

The function u_0 will solve

$$\begin{array}{ll} \Delta u_0(x) = \chi_{\{x;\, x\cdot\nu(z)<0\}} & \text{in } \mathbb{R}^n \\ u_0(x) = 0 & \text{on } \{x;\ x\cdot\nu(z) = 0 \\ \nabla u_0(0) = 0 & \text{on } \{x;\ x\cdot\nu(z) = 0, \end{array} \tag{6.5}$$

where the equation follows from (6.4) and the last two equations follows from $C^{1,\alpha}$ convergence.

By the Cauchy-Kovaleskaya theorem the system (6.5) has a unique solution which is easily verified to be the solution u_0 specified in (6.5). This finishes the sketch of the proof. □

Lemma 6.2. *Let* u^j *be a solution to the normalized obstacle problem in* $\mathcal{D}$*. Assume furthermore that* $B_3(0) \subset \mathcal{D}$*,* $0 \in \Gamma_{u^j}$*,* $u^j(x) = 0$ *for* $x_n < -1/2$ *and that*

$$\lim_{j\to\infty} \|\nabla' u^j\|_{L^2(B_1(0))} = 0. \tag{6.6}$$

Then $u^j \to \frac{1}{2}(x_n)_+^2$ *locally uniformly in* $B_1(0)$*. Furthermore, any sequence* $x^j \in \Gamma_{u^j} \cap B_r(0)$ *will converge uniformly to the set* $\{x_n = 0\}$ *for any* $r < 1$*.*

Sketch of the proof: Let us begin by showing that u^j converges. Since $0 \in \Gamma_{u^j}$ it follows from Theorem 5.1 that $|u^j(x)|$ is uniformly bounded in $B_{3/2}(0)$. Lemma 5.1 implies that u^j converges uniformly to some function u^0 (at least for some subsequence). The assumption (6.6) implies that $\nabla' u^0(x) = 0$. We draw the conclusion that $u^0(x)$ only depends on the x_n direction.

Next we want to know what differential equation u^0 solves. At any point y where $u^0(y) > 0$ there exists, by uniform convergence, a j_y such that $u^j(y) > 0$ for all $j > j_y$ and since the functions u^j are uniformly continuous there exists a small neighborhood around y where $u^j > 0$ (maybe for a larger j_y). It follows that $\Delta u^j = \chi_{\Omega_{u^j}} = 1$ in a neighborhood of y and therefore that $\Delta u^0(y) = 1$. We may conclude that $\Delta u^0 = \chi_{\Omega_{u^0}}$, which means that

$$\frac{\partial^2 u^0(x)}{\partial x_n^2} = \chi_{\Omega_{u^0}},$$

since u^0 only depend on the x_n-direction.

Moreover, since the convergence $u^j \to u^0$ is uniform and, by assumption, $u^j(0) = 0$ for all j it follows that $u^0(0) = 0$. And by non-degeneracy quadratic growth and uniform convergence it also follows that $Cr^2 > \sup_{B_r(0)} |u^0| > 0$ for any $r > 0$. This last statement clearly implies that $\frac{\partial u^0(0)}{\partial x_n} = 0$.

We have therefore shown that u^0 solves

$$\begin{array}{ll} \frac{\partial^2 u^0(x)}{\partial x_n^2} = \chi_{\Omega_{u^0}} & \text{in } B_1(0) \\ 0 < \sup_{B_r(0)} |u^0| < Cr^2 & \text{for all } 0 < r < 1/2 \\ \frac{\partial u^0(0)}{\partial x_n} = 0 & \\ u^0(x) = 0 & \text{for } x_n < -1/2 \end{array}$$

where the last line follows from uniform convergence and the assumption that $u^j(x) = 0$ for $x_n < -1/2$. We may conclude, by solving the ODE, that $u^0(x) = \frac{1}{2}(x_n)_+^2$.

Next we show that the free boundaries Γ_{u^j} converges uniformly to $\{x_n = 0\}$ we argue by contradiction. Clearly if $x^j \in \{x_n > 0\} \cap B_r(0)$ it follows that $x^j \to \{x_n = 0\}$ since

$$0 = u^j(x^j) \to \frac{1}{2}(x^j \cdot e_n)_+^2,$$

where the first equality follows from the assumption that $x^j \in \Gamma_{u^j}$. To show that any sequence $x \in \Gamma_{u^j} \cap B_r(0) \cap \{x_n < 0\}$ also converges to the line $\{x_n = 0\}$ we argue by contradiction. If not then there exists some real number $\tau > 0$ such that $x^j \cdot e_n < -\tau$. But then non-degeneracy implies that $\sup_{B_{\tau/2}(x^j)} |u^j| > c\tau^2$ which contradicts that $u^j \to 0$ uniformly in $\{x_n > 0\}$. □

7 Taylor's Theorem - some undergraduate analysis.

This section is a digression into undergraduate analysis. We will discuss several versions of Taylor's Theorem.

We will formulate our results in Hölder spaces and therefore begin by stating the definition of the space $C^{k,\alpha}(\mathcal{D})$.

Definition 7.1. *We will denote the space of all bounded continuous functions, equipped with the norm*

$$\|u\|_{C(\mathcal{D})} = \sup_{x \in \mathcal{D}} |u(x)|,$$

on a domain $\mathcal{D}$ by $C(\mathcal{D})$, or at times $C^0(\mathcal{D})$.

We say that a function, $u(x)$, defined on $\mathcal{D}$ is Hölder continuous with exponent $0 < \alpha < 1$ if

$$[u]_{C^\alpha(\mathcal{D})} = \sup_{x,y \in \mathcal{D}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty.$$

The space of all Hölder continuous functions with exponent $\alpha > 0$ defined on $\mathcal{D}$ will be denoted $C^\alpha(\mathcal{D})$. The space $C^\alpha(\mathcal{D})$ will be equipped with the norm

$$\|u\|_{C^\alpha(\mathcal{D})} = \sup_{x\in\mathcal{D}} |u(x)| + \sup_{x,y\in\mathcal{D}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} = \|u\|_{C(\mathcal{D})} + [u]_{C^\alpha(\mathcal{D})}. \tag{7.1}$$

The space of of $k-$times differentiable functions, $u(x)$, defined on $\mathcal{D}$ such that $D^l u(x) \in C^\alpha(\mathcal{D})$, for $l = 0, 1, 2, ..., k$ where D^l stands for the vector consisting of all partial derivatives of order l, with norm

$$\|u\|_{C^{k,\alpha}(\mathcal{D})} = \sum_{l=0}^{k} \|D^l u\|_{C(\mathcal{D})} + \left[D^k u\right]_{C^\alpha(\mathcal{D})} \tag{7.2}$$

will be denoted $C^{k,\alpha}(\mathcal{D})$.

Remark: *Often one writes $f \in C^1(\mathcal{D}) = C^{1,0}(\mathcal{D})$ if $f \in C(\mathcal{D})$ and $f' \in C(\mathcal{D})$, the norm $\|f\|_{C^1(\mathcal{D})} = \|f\|_{C(\mathcal{D})} + \|f'\|_{C(\mathcal{D})}$. The notation is somewhat confused here since $C^1(\mathcal{D})$ is not the same as $f \in C^\alpha(\mathcal{D})$ with $\alpha = 1$. For the space of Lipschitz functions, that is functions that satisfies (7.1) with $\alpha = 1$, one usually use the notation $f \in C^{0,1}(\mathcal{D})$.*

In (7.2) we interpret $D^l u$ as the vector consisting of all partial derivatives of order l.

We will now state a simple version of Taylor's Theorem in $C^{1,\alpha}(a, b)$.

Theorem 7.1. [TAYLOR'S THEOREM] *Let (a, b) be an intervall in $\mathbb{R}$ and assume that $f \in C^{1,\alpha}(a, b)$ and that $c \in (a, b)$. Then*

$$|f(x) - f(c) - f'(c)(x - c)| = B(x, c)|x - c|^{1+\alpha},$$

where $|B(x, c)| \leq [f']_{C^\alpha(a,b)}$.

Proof: The proof is absolutely standard and can be found in almost any book on first year calculus. We define

$$g(x) = f(x) - (f(c) + f'(c)(x - a)), \tag{7.3}$$

then $g(x) = B(x, c)(x - a)^{1+\alpha}$; therefore we need to show that

$$|g(x)| \leq \|f'\|_{C^\alpha(a,b)}|x - a|^{1+\alpha}.$$

Since $g(x)$ is C^1 (since $f(x)$ is) we may use the mean value theorem we can deduce that there exists a point ξ between a and x such that

$$g(x) - g(c) = (x - c)g'(\xi) \Rightarrow |g(x)| \leq |x - c||g'(\xi)|. \tag{7.4}$$

Using the definition of g, in particular that $g'(c) = 0$ by (7.3), and the assumption that $f \in C^{1,\alpha}$ we can conclude that

$$|g'(\xi)| = |g'(\xi) - g'(c)| = |f'(\xi) - f'(c)| \leq$$

$$\leq \frac{|f'(\xi) - f'(c)|}{|\xi - c|^\alpha}|\xi - c|^\alpha \leq [f']_{C^\alpha(a,b)}|\xi - c|^\alpha.$$

Inserting this final estimate in (7.4), and taking into consideration that $|\xi - c| < |x - c|$ since ξ is between x and c, we conclude that

$$|g(x)| \leq |x - c||g'(\xi)| \leq [f']_{C^\alpha(a,b)}|x - c|^{1+\alpha}.$$

□

For our purposes Taylor's Theorem is not that useful. We want to prove $C^{1,\alpha}$, but Taylor's Theorem assumes $C^{1,\alpha}$. Interestingly, and of extreme importance for regularity theory for partial differential equations, one can reverse Taylor's theorem and use the conclusion to prove $C^{1,\alpha}$

Lemma 7.1. [Reverse Taylor Theorem] *Let $f(x) \in C(a,b)$ be a differentiable function with uniformly bounded derivatives. Assume furthermore that there exists a constant C_0 such that for any $y \in (a,b)$ and $r > 0$*

$$\sup_{|y-x|\leq r} |f(x) - f(y) - f'(y)(x-y)| \leq C_0 r^{1+\alpha}. \tag{7.5}$$

Then $f(x) \in C^{1,\alpha}(a,b)$ and $f(x)$ satisfies the following estimates

$$[f'(x)]_{C^\alpha(a,b)} \leq 2C_0 \tag{7.6}$$

and

$$\|f\|_{C^{1+\alpha}(a,b)} \leq \|f\|_{C^1(a,b)} + 2C_0. \tag{7.7}$$

Proof: To write an analytical proof of this theorem is not very illustrative. We will therefore argue informally from a graph. Once the general idea is understood the reader is invited to fill in the details.

We begin by noticing that it is enough to prove (7.6) since by definition

$$\|f\|_{C^{1+\alpha}(a,b)} \leq \|f\|_{C^1(a,b)} + [f]_{C^\alpha(a,b)},$$

and therefore (7.7) is directly implied by (7.6).

In order to show (7.6) we need to show that for any $x, y \in (a,b)$

$$|f'(x) - f'(y)| \leq 2C_0|x - y|^\alpha. \tag{7.8}$$

There is no loss of generality to assume that $f(y) = f'(y) = 0$ since (7.8) is unchanged of we add a linear function to f. Furthermore we may, without loss of generality, assume that $y = 0$ and that $x > 0$.

With these simplifying assumptions we need to show that $f'(x) \leq 2C_0|x|^\alpha$. We will argue by a worst case scenario. So let us set $r = x > 0$. Then the worst case scenario is that $f(x) = C_0 r^{1+\alpha}$, since $|f(x)| \leq C_0 r^{1+\alpha}$ by (7.5). Also, by (7.5) with the roles of x and $y = 0$ reversed, we know that

$$C_0 r^{1+\alpha} \geq f(y) - f(x) - f'(x)(y - x) = -C_0 r^{1+\alpha} + f'(x) r$$
$$\Rightarrow f'(x) \leq 2C_0 r^\alpha = 2C_0|x - y|^\alpha.$$

One argues similarly, using a worst case scenario with $f(x) = -C_0 r^{1+\alpha}$, to estimate $f'(x) \geq -2C_0|x - y|^\alpha$. We leave the details to the reader. □

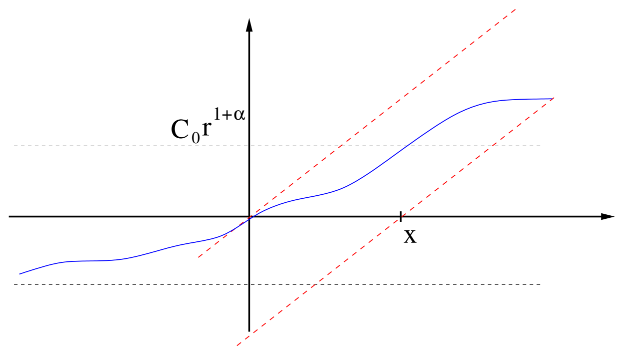


Figure: *The above figure shows the main idea of the argument. If $f(0) = 0$ and $f'(0) = 0$ then the graph has to be contained in a strip (dashed black) of with $2C_0 r^{1+\alpha}$ for all $|x| \leq r$. If we consider the point $x = r$ then the graph also has to be trapped between a strip of height $2C_0 r^{1+\alpha}$ centered at $f(x)$. This strip, dashed red in the figure, has the same direction as $f'(x)$. This implies that if $f'(x)$ is to large, larger than the worst case as indicated in the figure, then the point $(y, f(y))$ is not contained in the strip as it should be by assumption (7.19). This gives the desired estimate on $f'(x)$.*

Lemma 7.1 still assumes that $f(x) \in C^1$. In working with free boundary problems we do not even know that about the free boundary so we need to weaken the assumptions further. The next lemma significantly weakens the assumptions need in order to prove $C^{1,\alpha}-$regularity. Notice in particular that we do not assume that $f(x)$ is C^1 but instead assume that f has some approximate tangent line $l_{y,\delta}(x - y) + f(y)$.

Lemma 7.2. [REVERSE TAYLOR THEOREM WITH APPROXIMATE NORMAL.] *Let $f(x)$ be a function defined on $(-1, 1)$ assume furthermore that there exists a constant c_0 such that for every $y \in (a, b)$ and $r > 0$ there exists an $l_{y,r} \in \mathbb{R}$ such that*

$$\sup_{|y-x|\leq r} |f(x) - f(y) - l_{y,r}(x - y)| \leq c_0 r^{1+\alpha}, \tag{7.9}$$

for some $0 < \alpha < 1$.

Then $f \in C^{1,\alpha}(-1, 1)$

$$[f']_{C^\alpha(-1,1)} \leq Cc_0, \tag{7.10}$$

where C only depends on α.

Proof: The proof, which entirely depends on undergraduate calculus, consists of estimate the difference between $l_{y,2^{-k}r}$ and $l_{y,2^{-k-1}r}$. If $|l_{y,2^{-k}r} - l_{y,2^{-k-1}r}|$ is small enough, as a matter of fact the difference will form a Cauchy sequence, then we can consider the limit $\lim_{k\to\infty} l_{y,2^{-k}r}$. Once we have identified a limit it will be rather easy to reduce this Lemma to the previous one. We proceed in several steps.

Step 1. *We have the following estimate for the difference*

$$|l_{y,2^{-k}r} - l_{y,2^{-k-1}r}| \leq 3c_0 \left(2^{-k}r\right)^{\alpha}.$$

Proof of Step 1: The proof is very similar to the proof of Lemma 7.1. We know, assumption (7.9), that

$$\sup_{|y-x|\leq 2^{-k}r} \left|f(x) - f(y) - l_{y,2^{-k}r}(x-y)\right| \leq c_0(2^{-k}r)^{1+\alpha} \tag{7.11}$$

and that

$$\sup_{|y-x|\leq 2^{-k-1}r} \left|f(x) - f(y) - l_{y,2^{-k-1}r}(x-y)\right| \leq c_0(2^{-k-1}r)^{1+\alpha}. \tag{7.12}$$

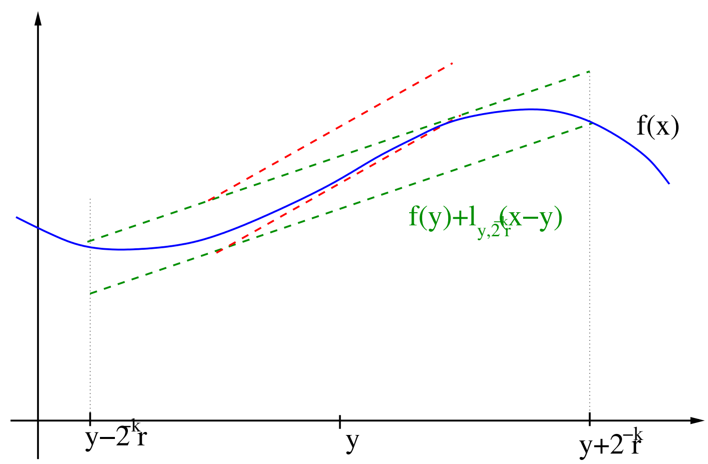


Figure: *The geometry of the proof of step 1. The graph of f is trapped in two strips: the dashed red and the dashed green. In order for this to be possible the slope of the red lines can not be to steep, that is $l_{y,2^{-k-1}}$ can not be to large.*

From (7.11) and (7.12) we can conclude, with $x = y + 2^{-k-1}r$, that

$$f(y) + l_{y,2^{-k-1}r}2^{-k-1}r - c_0(2^{-k-1}r)^{1+\alpha} \leq f(y + 2^{-k}r) \leq$$

$$\leq f(y) + l_{y,2^{-k}r}2^{-k-1}r + c_0(2^{-k}r)^{1+\alpha},$$

which after rearrangement of terms leads to

$$l_{y,2^{-k-1}r} - l_{y,2^{-k}r} \leq c_0\left(\left(2^{-k-1}r\right)^{\alpha} + 2\left(2^{-k}r\right)^{\alpha}\right) \leq 3c_0\left(2^{-k}r\right)^{\alpha}. \tag{7.13}$$

Arguing similarly from (7.11) and (7.12) we can conclude with $x = y - 2^{-k-1}r$ we can conclude that

$$l_{y,2^{-k-1}r} - l_{y,2^{-k}r} \geq -3c_0\left(2^{-k}r\right)^{\alpha}. \tag{7.14}$$

From (7.13) and (7.14) we conclude that

$$|l_{y,2^{-k-1}r} - l_{y,2^{-k}r}| \leq 3c_0 \left(2^{-k}r\right)^{\alpha}. \tag{7.15}$$

This proves step 1.

Step 2. *For each $y \in (-1,1)$ the limit $\lim_{k\to\infty} l_{y,2^{-k}r} = l_y$ exists. Furthermore*

$$|l_y - l_{y,2^{-k}r}| \leq 3c_0 r^{\alpha} \frac{2^{-\alpha(k+1)}}{1-2^{-\alpha}}.$$

Proof of step 2: We notice that the sequence $a_k = l_{y,2^{-k}r}$ is a Cauchy sequence since, for $m > k$,

$$|a_m - a_k| = \left|\sum_{j=k+1}^{m} (a_{j+1} - a_j)\right| \leq \left\{ \begin{array}{c} \text{using} \\ \text{step 1} \end{array} \right\} \leq$$

$$\leq 3c_0 \sum_{j=k+1}^{m} \left(2^{-j}r\right)^{\alpha} = 3c_0 r^{\alpha} \sum_{j=k+1}^{m} \left(2^{-\alpha}\right)^{j} = \tag{7.16}$$

$$= 3c_0 r^{\alpha} \frac{2^{-\alpha(k+1)} - 2^{-\alpha(m+1)}}{1-2^{-\alpha}} \leq 3c_0 r^{\alpha} \frac{2^{-\alpha(k+1)}}{1-2^{-\alpha}}$$

Clearly the right hand side tends to zero as $k \to \infty$, it follows that $a_k = l_{y,2^{-k}r}$ is a Cauchy sequence and therefore convergent.

To derive (7.15) we only need to pass to the limit $m \to \infty$ in (7.16); then $a_m \to l_y$ on the left side whereas the right side is preserved.

Step 3: *The following estimate holds*

$$\sup_{|y-x|\leq r} |f(x) - f(y) - l_y(x-y)| \leq Cc_0 r^{1+\alpha}, \tag{7.17}$$

where C depend on $\alpha > 0$ but is independent of y and c_0.

In particular, f is differentiable and, by Lemma 7.1, satisfies (7.20).

Proof of step 3. The conclusion of the step follows directly form step 2 and the triangle inequality. Let us provide some details,

$$\sup_{|y-x|\leq r} |f(x) - f(y) - l_y(x-y)| \leq \tag{7.18}$$

$$\leq \sup_{|y-x|\leq 2^{-k}} \left|f(x) - f(y) - l_{y,2^{-k}}(x-y) + (l_{y,2^{-k}}(x-y) - l_y(x-y))\right|,$$

where we choose k as the largest constant such that $r \leq 2^{-k}$.

Using the triangle inequality in (7.18) we continue to estimate

$$\sup_{|y-x|\leq r} |f(x) - f(y) - l_y(x-y)| \leq$$

$$\leq \sup_{|y-x|\leq 2^{-k}} \left|f(x)-f(y)-l_{y,2^{-k}}(x-y)\right| +$$
$$+ \sup_{|y-x|\leq 2^{-k}} \left|l_{y,2^{-k}}(x-y)-l_y(x-y))\right| \leq$$
$$\leq c_0(2^{-k})^{1+\alpha} + 3c_0\frac{\left(2^{-k}\right)^{1+\alpha}}{1-2^{-\alpha}} \leq Cc_0(2^{-k})^{1+\alpha} \leq 2^{1+\alpha}Cc_0 r^{1+\alpha},$$

where we used, in the last inequality, that $2^{-k}\leq 2r$ since k was the smallest constant such that $r\leq 2^{-k}$. This proves (7.17).

It follows immediately from (7.17) that f is differentiable. This proves that f satisfies the assumptions of Lemma 7.1 and we may conclude from that lemma that (7.10) holds. □

Even though we prove the above Lemma in $\mathbb{R}$ there is nothing in the proof that really uses that $\mathbb{R}$ is one dimensional (except that we use some graphs to illustrate the proofs). We may therefore state the same result in $\mathbb{R}^n$.

Corollary 7.1. *Let $u(x)$ be a function defined on $B_1(0)$, assume furthermore that there exists a constant c_0 such that for any $y\in B_1(0)$ and $r>0$ there exists an approximate normal vector $l_{y,r}\in\mathbb{R}^n$ such that*

$$\sup_{|y-x|\leq r} |u(x)-u(y)-l_{y,r}\cdot(x-y)| \leq c_0 r^{1+\alpha}, \tag{7.19}$$

for some $0<\alpha<1$.

Then $u\in C^{1,\alpha}(B_1(0))$ and

$$[\nabla u]_{C^\alpha(B_1(0))} \leq Cc_0, \tag{7.20}$$

where C only depends on α.

Idea of the proof: Notice that if we restrict u to the any line, $y+t\eta$ for $\eta\in\mathbb{R}^n$, through y then $u(y+t\eta)$ becomes a function defined on $\mathbb{R}$. We may thus use the one dimensional proof. □

Let us finish this section with a simple, but extremely important, Theorem.

Theorem 7.2. [De Giorgi iterations] *Assume that $u(x)$ is a function defined on $B_1(0)$ and that $u(x)$ has the following property: There exists a $\delta_0>0$ and two constants $0<s,\kappa<1$ such that if there exists a vector $l_{y,r}\in\mathbb{R}^n$ such that*

$$\sup_{B_r(y)} |u(x)-u(y)-l_{y,r}\cdot(x-y)| \leq \delta r \tag{7.21}$$

for some $\delta<\delta_0$ then there exists a vector $l_{y,sr}\in\mathbb{R}$

$$\sup_{B_{sr}(y)} |u(x)-u(y)-l_{y,sr}\cdot(x-y)| \leq \delta\kappa sr. \tag{7.22}$$

It follows that if (7.21) is satisfied, with $4\delta<\delta_0$, then $u\in C^{1,\alpha}(B_{r/2}(y))$ for some α that depends only on s and κ.

Proof: The proof is a direct consequence of the previous analysis in this section. Notice that if (7.21) holds in $B_r(y)$ then, by (7.22), it holds in $B_{sr}(y)$ with $\kappa\delta$ in place of δ. We may iterate this and conclude that

$$\sup_{B_r(y)} |u(x) - u(y) - l_{y,r} \cdot (x-y)| \leq \delta r \tag{7.23}$$

$$\Rightarrow \sup_{B_{s^k r}(y)} |u(x) - u(y) - l_{y,s^k r} \cdot (x-y)| \leq \delta \kappa^k s^k r.$$

Moreover, if (7.21) holds in $B_r(y)$ and $z \in B_{r/2}(y)$. Then we know that there is a plane, namely $u(y) + l_{y,r} \cdot (x-y)$, that approximates $u(x)$ well in $B_{r/2}(z)$. In particular, if $\mathcal{P}$ denotes the set of all planar functions

$$\mathcal{P} = \{a + l \cdot x;\ a \in \mathbb{R} \text{ and } l \in \mathbb{R}^n\}$$

then, since $u(y) + l_{y,r} \cdot (x-y) \in \mathcal{P}$,

$$\inf_{p\in\mathcal{P}} \left(\sup_{x\in B_{r/2}(z)} |u(x) - p(x)| \right) \leq \sup_{x\in B_{r/2}(z)} |u(x) - u(y) + l_{y,r} \cdot (x-y)| \leq \delta r. \tag{7.24}$$

We can conclude that, with $p \in \mathcal{P}$ being the minimizer in (7.24),

$$\inf_{l\in\mathbb{R}^n} \left(\sup_{x\in B_{r/2}(z)} |u(x) - u(z) - l \cdot (x-z)| \right) =$$

$$= \inf_{l\in\mathbb{R}^n} \left(\sup_{x\in B_{r/2}(z)} |u(x) - p(x) - (u(z) - p(x)) - l \cdot (x-z)| \right) \leq \tag{7.25}$$

$$\leq \sup_{x\in B_{r/2}(z)} |u(x) - p(x)| + |u(z) - p(z)| \leq 4\left(\frac{r}{2}\right)\delta,$$

where we used (7.24) in the last inequality.

If $\delta < \delta_0/4$ then (7.25) implies that there exists an $l_{z,r} \in \mathbb{R}^n$ such that

$$\sup_{B_{r/2}(z)} |u(x) - u(z) - l_{z,r} \cdot (x-z)| \leq 4\left(\frac{r}{2}\right)\delta < \delta_0 \left(\frac{r}{2}\right), \tag{7.26}$$

that is (7.21) is satisfied in the ball $B_{r/2}(z)$. We may conclude, as in (7.23), that

$$\sup_{B_{s^k r/2}(z)} |u(x) - u(z) - l_{z,s^k r/2} \cdot (x-z)| \leq 4\delta\kappa^k s^k \left(\frac{r}{2}\right), \tag{7.27}$$

for any point $z \in B_{r/2}(y)$.

Notice that if $\alpha = \frac{\ln(\kappa)}{\ln(s)}$ then

$$4\delta\kappa^k s^k \left(\frac{r}{2}\right) = \frac{4\delta}{r^\alpha}\left(\frac{s^k r}{2}\right)^{1+\alpha} = c_0 \left(\frac{s^k r}{2}\right)^{1+\alpha}, \tag{7.28}$$

with $c_0 = \frac{4\delta}{r^\alpha}$ is a constant.

Observe that (7.27) together with (7.28) implies that,

$$\sup_{B_{s^k r/2}(z)} |u(x) - u(z) - l_{z,s^k r/2} \cdot (x-z)| \le c_0 \left(\frac{s^k r}{2}\right)^{1+\alpha}, \tag{7.29}$$

notice that this is just (7.19).[34] The conclusion of the Theorem follows from Lemma 7.2. □

8 Regularity of Free boundaries at Flat Points.

In the last section we investigated several reverse versions of the Taylor Theorem that helped us to show that different functions are $C^{1,\alpha}$. In this section we will use use reverse Taylor Theorem to prove that the free boundary of the obstacle problem is $C^{1,\alpha}$ under certain assumptions. In order to do that we need to find a good, and different, way to characterize approximate tangent planes; we do not even know that the free boundary is a graph so we can not, directly, use the approach from the last section that was entirely based on the assumption that the object whose regularity we tried to prove was a graph.

We will however follow the outline in the last section - but in reverse. The first order of business will be to identify a good candidate for the approximate normal of the free boundary; we will use the gradient of the solution u for that. Then, working in reverse compared to the previous section, we will derive the improvement from (7.21) to (7.22) in the De Giorgi iteration method (Theorem 7.2). This will be done in Lemma 8.1, Lemma 8.2 and Proposition 8.1. Thereafter we will mimic Lemma 7.2 in Proposition 8.2 and its corollaries. We end the section with Theorem 8.1 which follows the proof of Lemma 7.1.

Theorem 8.1 is the main free boundary regularity theorem which states that if the solution u to the obstacle problem is close to a half-space solution $\frac{1}{2}(x_n)_+^2$ in a ball $B_1(0)$ then the free boundary is a $C^{1,\alpha}$ graph in $B_{1/2}(0)$. We also provide estimates of the $C^{1,\alpha}-$norm of the free boundary (which are new). Using Theorem 8.1 together with the fact that the free boundary has a measure theoretic normal at a.e. point it follows that the free boundary is $C^{1,\alpha}$ in a neighborhood of a.e. point: see Corollary 8.3

We begin by identifying a natural approximate normal of the free boundary. For this we will use the gradient of the solution of the obstacle problem. If we have a solution, $u(x)$, to the obstacle problem that is close to the half-space solution $\frac{1}{2}(x_n)_+^2$ then we would want the approximate normal to point in the $-e_n$ direction. That means that a good candidate for an approximate normal,

[34]There is a slight difference between (7.29) and (7.19). In (7.29) we only allow a discrete set of radii. This is a very slight problem that we encountered earlier, see the proof of step 3 in Lemma 7.2. We encourage the reader to fill the gap (it can be done as in step 3).

η, to the free boundary is

$$\eta = -\frac{\int_{B_1(0)} \nabla u(x)dx}{\left|\int_{B_1(0)} \nabla u(x)dx\right|} \approx -e_n. \tag{8.1}$$

We may rotate the coordinate system so that the approximate normal $\eta = -e_n$.

Definition 8.1. *We say that a coordinate system is normalized with respect to a solution u to the obstacle problem if*

$$\|\nabla' u\|_{L^2(B_1(0))} \leq \inf_\eta \|\nabla u - \eta(\eta \cdot \nabla u)\|_{L^2(B_1(0))},$$

where the infimum is taken over all unit vectors η.

Notice that if we have a coordinate system that is normalized with respect to u then the average gradient of u has to point in the e_n direction. Furthermore, if $u = \frac{1}{2}(x_n)_+^2$ then

$$\|\nabla' u\|_{L^2(B_1(0)} = 0.$$

Therefore, the norm $\|\nabla' u\|_{L^2(B_1(0)}$, where $\nabla' u = (\partial_1 u, \partial_2 u, ..., \partial_{n-1} u, 0)$, provides a good estimate on how close the solution is to being a half-space solution (see Lemma 6.2); and also a good estimate on how close the approximate normal η is to being a real normal. Theorem 7.2 essentially states that if we can prove an improvement in the approximation of the normal when we consider a smaller ball then we can prove regularity of the free boundary.

In order to get some improvement of the approximate normal (as in the improvement from (7.21) to (7.22)) we need to show that our solution u, or in this case the derivatives of u, has some good properties. The first step is to show that the derivatives of u behaves like harmonic functions (which we know are good in the sense of Lemma 8.2), at least when u is close to a half space solution.

Lemma 8.1. *Let $\epsilon_j \to 0$ and u^j be a sequence of minimizers to the obstacle problem in $B_3(0)$ and*

1. *the coordinate system is normalized with respect to u^j for each j*
2. $0 \in \Gamma_{u^j}$,
3. $u^j(x) = 0$ *in the set* $\{x \in B_1(0);\ x_n < -1/2\}$ *and that*
4. $\|\nabla' u^j\|_{L^2(B_1(0))} = \epsilon_j \to 0$.

Furthermore we let

$$v_i^j(x) = \frac{1}{\epsilon_j}\frac{\partial u^j(x)}{\partial x_i} \quad \textit{for } i = 1, 2, ..., n-1.$$

Then for any fixed $0 < r < 1$ there is a subsequence of j, that we will continue to denote by j, such that $v_i^j \to v_i^0$ strongly in $L^2(B_r(0))$, weakly in $L^2(B_1(0))$ and weakly in $W^{1,2}(B_r(0))$. Furthermore v_i^0 satisfies

$$\begin{aligned} &\Delta v_i^0 = 0 \quad && \text{in } B_1^+(0) \\ &v_i^0 = 0 && \text{in } B_1^-(0). \end{aligned} \tag{8.2}$$

The last line in the above equation, together with $v_i^0 \in W^{1,2}(B_r(0))$ implies that $v_i^0(x', 0) = 0$.

Proof: Notice that, since $\Delta u^j = 1$ in Ω_{u^j}, $\Delta v_i^j = 0$ in the set Ω_{u^j}. Also since, for any $\delta > 0$, $u^j = 0$ in $\{x_n < -\delta\}$ for j large enough it follows that $v_i^j = 0$ in $\{x_n < -\delta\}$ for j large enough. We can thus conclude that (8.2) holds if $v_i^j \to v_i^0$. It remains to prove the convergence.

Since, by Theorem 4.1, $u^j \in W^{2,2}(B_{5/4}(0))$ it follows that $v_i^j \in W^{1,2}(B_{5/4})$. Using Theorem 4.1 we can even get the estimate, for each $r < 1$,

$$\int_{B_r(0)} |\nabla v_i^j(x)|^2 dx = \frac{1}{\epsilon_j^2} \int_{B_r(0)} \left| \nabla \frac{\partial u^j}{\partial x_i} \right|^2 dx \leq$$

$$\leq \frac{C_r}{\epsilon_j^2} \int_{B_1(0)} \left| \frac{\partial u^j}{\partial x_i} \right|^2 dx \leq C_r.$$

We can therefore conclude that $v_i^j \in W^{1,2}(B_r(0))$ for any $0 < r < 1$. Therefore, by the weak compactness in $W^{1,2}(B_r(0))$, see Theorem 2.9, it follows that $v_i^j \rightharpoonup v_i^0$ in $W^{1,2}(B_r(0))$. Furthermore, by the compactness of the embedding $W^{1,2} \to L^2$ (see Theorem 2.9), it follows that $v_i^j \to v_i^0$ strongly in $L^2(B_r(0))$ for every $r < 1$.

That $v_i^0(x', 0) = 0$ follows from an argument exactly as in the trace Theorem. This proves the Lemma. □

In the following lemma we derive a regularity improvement property of harmonic functions. Since v_i^j converges to harmonic functions this property will be inherited by v_i^j for large j which will help us to show that u also improves in smaller balls.

Lemma 8.2. *Let v_i^0 be as in Lemma 8.1. Then we may write*

$$v_i^0(x) = \sum_{k=2}^{\infty} p_{k,i}(x), \tag{8.3}$$

where $p_{k,i}(x)$ are harmonic that are homogeneous of order k and orthogonal in the L^2-norm:

$$\int_{B_1^+(0)} p_{k,i}(x) p_{l,i}(x) dx = 0 \text{ if } k \neq l.$$

Proof: Since v_i^0 is harmonic with zero boundary data om $\{x_n = 0\}$ it follows from classical theory for harmonic functions that we can expand v_i^0 as a power-series

$$v_i^0(x) = \sum_{k=1}^{\infty} p_{k,i}(x) \tag{8.4}$$

where $p_{k,i}(x)$ are harmonic polynomials of degree k - the polynomials are also orthogonal with respect to the L^2 norm. What we need to prove in order to prove (8.3) is that the polynomial $p_{1,i}(x)$ in (8.4) is identically equal to zero. This is a very important property since, as we will see later, it is the fact that $p_{1,i}(x) = 0$ that will give us the important gain we need in order to prove regularity of the free boundary.

So far we have not used the assumption that the coordinate system is normalized with respect to u^j. This means, see Definition 8.1, that for any unit vector η_j

$$\|\nabla' u^j\|^2_{L^2(B_1(0))} \leq \|\nabla u^j - \eta_j(\eta_j \cdot \nabla u^j)\|^2_{L^2(B_1(0))}. \tag{8.5}$$

Notice that if we divide the left side of (8.5) by ϵ_j^2 then, since $v_i^j = \frac{1}{\epsilon_j}\frac{\partial u^j}{\partial x_i}$, it will converge to

$$\sum_{i=1}^{n-1} \|v_i^0\|^2_{L^2(B_1(0)} = \sum_{i=1}^{n-1}\sum_{k=1}^{\infty} \|p_{k,i}\|^2_{L^2(B_1(0)}, \tag{8.6}$$

where we used that the polynomials $p_{k,i}$ are orthogonal.

Since $p_{1,i}(x)$ are harmonic and homogeneous of order 1 and vanish on $x_n = 0$ it follows that $p_{1,i} = a_i x_n$. We need to show that $a_i = 0$. If $a_i \neq 0$ then we may choose the vector

$$\eta_j = \frac{(\epsilon_j a_1, \epsilon_j a_2, ..., \epsilon_j a_{n-1}, 1)}{\left(1 + \epsilon_j^2 \sum_{i=1}^{n-1} a_i^2\right)^{1/2}} = \tau_j(\epsilon_j a_1, \epsilon_j a_2, ..., \epsilon_j a_{n-1}, 1),$$

where $\tau_j = \frac{1}{\left(1+\epsilon_j^2 \sum_{i=1}^{n-1} a_i^2\right)^{1/2}} = 1 + O(\epsilon_j^2)$. With this notation the right side of (8.5) can be written, after dividing by ϵ_j^2,

$$\frac{1}{\epsilon_j^2}\|\nabla u - \eta_j(\eta_j \cdot \nabla u^j)\|^2_{L^2(B_1(0))} =$$

$$= \sum_{i=1}^{n-1} \left\| \frac{1}{\epsilon_j}\frac{\partial u^j}{\partial x_i} - \tau_j a_i \left(\eta_j \cdot \nabla u^j\right) \right\|^2_{L^2(B_1)} + \frac{1}{\epsilon_j^2}\left\| \frac{\partial u^j}{\partial x_n} - \tau_j \left(\eta_j \cdot \nabla u^j\right)\right\|^2_{L^2(B_1)} =$$
$$= \sum_{i=1}^{n-1} \left\| v_i^j - \tau_j a_i \left(\eta_j \cdot \nabla u^j\right) \right\|^2_{L^2(B_1(0))} + \frac{1}{\epsilon_j^2}\left\| \frac{\partial u^j}{\partial x_n} - \tau_j \left(\eta_j \cdot \nabla u^j\right)\right\|^2_{L^2(B_1(0))}. \tag{8.7}$$

In order to continue to estimate (8.7) we notice that

$$\eta_j \cdot \nabla u^j = \underbrace{\tau_j \epsilon_j \sum_{i=1}^{n-1} a_i v_i^j}_{=O(\epsilon_j)} + \tau_j^2 \underbrace{\frac{\partial u^j}{\partial x_n}}_{\to x_n} \to x_n, \tag{8.8}$$

where we used $\frac{\partial u^j}{\partial x_i} = \epsilon_j v_i^j$ and, by Lemma 6.2, $u^j \to \frac{1}{2}(x_n)_+^2$ in $W^{1,2}(B_1)$, and that

$$\frac{\partial u^j}{\partial x_n} - \tau_j \left(\eta_j \cdot \nabla u^j\right) = (1-\tau_j^2)\frac{\partial u^j}{\partial x_n} + \tau_j \epsilon_j^2 \sum_{i=1}^{n-1} a_i v_i^j = O(\epsilon_j^2), \tag{8.9}$$

where we also used that $\tau_j = 1 + O(\epsilon_j^2)$.

Using (8.8) and (8.9) together with $\tau_j \to 1$ in (8.7) we can conclude that

$$\frac{1}{\epsilon_j^2}\|\nabla u - \eta_j(\eta_j \cdot \nabla u^j)\|^2_{L^2(B_1(0))} =$$

$$= \sum_{i=1}^{n-1} \left\| v_i^j - \tau_j^2 a_i x_n \right\|^2_{L^2(B_1(0))} + o(1) \to \tag{8.10}$$

$$\to \underbrace{\sum_{i=1}^{n-1} \|p_{1,i} - a_i x_n\|^2_{L^2(B_1(0)}}_{=0} + \sum_{i=1}^{n-1}\sum_{k=2}^{\infty} \|p_{k,i}\|^2_{L^2(B_1(0)},$$

where we used that $p_{1,i} = a_i x_n$ to conclude that the first sum on the last line is zero.

Comparing the expression of the norm in (8.6) with the expression in (8.10) we see that for ϵ_j small enough we get that

$$\frac{1}{\epsilon_j^2}\|\nabla u - \eta_j(\eta_j \cdot \nabla u^j)\|^2_{L^2(B_1(0))} \to \sum_{i=1}^{n-1}\sum_{k=2}^{\infty} \|p_{k,i}\|^2_{L^2(B_1(0)} <$$

$$< \sum_{i=1}^{n-1}\sum_{k=1}^{\infty} \|p_{k,i}\|^2_{L^2(B_1(0)} \leftarrow \frac{1}{\epsilon_j^2}\|\nabla' u^j\|^2_{L^2(B_1(0))},$$

contradiction the assumption that the coordinate system is normalized with respect to u^j, see (8.5). □

We are now ready to prove the main improvement of the approximation of the normal. The next proposition should be compared to (7.21)-(7.22) in the De Giorgi iteration Theorem 7.2.

Proposition 8.1. *For each $1/2 < \kappa < 1$ there exist an $\epsilon_\kappa > 0$ such that if u is a of minimizer to the obstacle problem in $B_3(0)$ and*

1. *the coordinate system is normalized with respect to u*

2. $0 \in \Gamma_u$,

3. $u(x) = 0$ *in the set* $\{x \in B_1(0);\ x_n < -1/2\}$ *and that*

4. $\|\nabla' u\|_{L^2(B_1(0))} = \epsilon < \epsilon_\kappa$.

Then

$$\|\nabla' u\|_{L^2(B_{1/2}(0))} \le \frac{\kappa}{2^{\frac{n+2}{2}}} \|\nabla' u\|_{L^2(B_1(0))}. \tag{8.11}$$

Proof: We will argue indirectly and assume that the proposition does not hold. Then there exists a $1/2 < \kappa < 1$ and a sequence of solutions to the obstacle problem u^j in $B_3(0)$ and a sequence $\epsilon_j \to 0$ such that u^j satisfies the assumptions of the proposition with $\epsilon = \epsilon_j$ and

$$\left\|\nabla' u^j\right\|_{L^2(B_{1/2}(0))} > \frac{\kappa}{2^{\frac{n+2}{2}}} \left\|\nabla' u^j\right\|_{L^2(B_1(0))}. \tag{8.12}$$

By Theorem 5.1 we know that there exists a constant C, depending only on the dimension, such that $\sup_{B_{3/2}} |u^j| \le C$. We may thus pick a subsequence, that we will assume to be the full sequence, of u^j such that $u^j \to u^0$ weakly in $B_{3/2}(0)$ as $j \to \infty$.

Since $\|\nabla' u^j\|_{L^2(B_1(0))} = \epsilon_j \to 0$ we can conclude that $\nabla' u^0 = 0$, we may conclude, from Lemma 6.2, that $u^0(x) = \frac{1}{2}(x_n)_+^2$. Furthermore the free boundary, Γ_{u^j}, converges uniformly to $\{x_n = 0\}$.

We would like to transfer this information about the limit u^0 back to the functions u^j, for large j. We will do that by using Lemma 8.1 and Lemma 8.2. Therefore, as in Lemma 8.1, we define

$$v_i^j(x) = \frac{1}{\epsilon_j} \frac{\partial u^j(x)}{\partial x_i} \qquad \text{for } i = 1, 2, ..., n-1.$$

We may conclude, from Lemma 8.1, that $v_i^j \to v_i^0$, and, from Lemma 8.2, that

$$\int_{B_1(0)} |v_i^0(x)|^2 dx = \int_{B_1(0)} \left| \sum_{k=2}^{\infty} p_k(x) \right|^2 dx.$$

The proof of the proposition is based on the following calculation

$$\int_{B_{1/2}(0)} |v_i^0(x)|^2 dx = \int_{B_{1/2}(0)} \left| \sum_{k=2}^{\infty} p_k(x) \right|^2 dx =$$

$$= \left\{ \begin{array}{l} \text{orthognlity} \\ \text{for } p_k \end{array} \right\} = \sum_{k=2}^{\infty} \int_{B_{1/2}(0)} |p_k(x)|^2 \, dx =$$

$$= \sum_{k=2}^{\infty} \frac{1}{2^n} \int_{B_1(0)} \left|p_k(2^{-1}x)\right|^2 dx =$$

$$= \left\{ \begin{array}{l} \text{homogenuity} \\ \text{for } p_k \end{array} \right\} = \frac{1}{2^n} \sum_{k=2}^{\infty} 2^{-2k} \int_{B_1(0)} |p_k(x)|^2 \, dx \le$$

$$\le \frac{1}{2^4} \frac{1}{2^n} \sum_{k=2}^{\infty} \int_{B_1(0)} |p_k(x)|^2 \, dx =$$

$$= \frac{1}{2^4} \frac{1}{2^n} \int_{B_1(0)} |v_i^0(x)|^2 dx.$$

Thus

$$\|v_i^0\|^2_{L^2(B_{1/2}(0))} \le \frac{1}{2^4} \frac{1}{2^n} \|v_i^0\|^2_{L^2(B_1(0))}. \tag{8.13}$$

Remembering, Lemma 8.1, that $v_0^j \rightharpoonup v_i^0$ and that

$$\sum_{i=1}^{n-1} \|v_i^j\|^2_{L^2(B_1(0))} = \frac{1}{\epsilon_j^2} \|\nabla' u^j\|^2_{L^2(B_1(0))} = 1$$

we can conclude, from Lemma 2.1, that $\sum_{i=1}^{n-1} \|v_i^0\|^2_{L^2(B_1(0))} \le 1$ and we can thus write (8.13)

$$\sum_{i=1}^{n-1} \|v_i^0\|^2_{L^2(B_{1/2}(0))} \le \frac{1}{2^4} \frac{1}{2^n}. \tag{8.14}$$

Since the convergence $v_i^j \to v_i^0$ is strong in $L^2(B_{1/2})$, by Lemma 2.9, we can conclude, that for j large enough (or ϵ_j small enough, say smaller than some ϵ_κ)

$$\sum_{i=1}^{n-1} \|v_i^j\|^2_{L^2(B_{1/2}(0))} \le \frac{\kappa^2}{2^2} \frac{1}{2^n} \tag{8.15}$$

The inequality (8.15) can be written for u^j as follows

$$\epsilon_j < \epsilon_\kappa \Rightarrow \|\nabla' u^j\|^2_{L^2(B_{1/2}(0))} = \epsilon_j^2 \sum_{i=1}^{n-1} \|v_i^j\|^2_{L^2(1/2)} \le$$

$$\le \frac{\kappa^2}{2^2} \frac{\epsilon_j^2}{2^n} = \frac{\kappa^2}{2^{n+2}} \|\nabla' u^j\|^2_{L^2(B_1(0))}.$$

Taking square roots in the last inequality finishes the proof. □

We will now prove some results corresponding to Lemma 7.2. We will split the proof into several results. The next Lemma loosely corresponds to step 1 in the proof of Lemma 7.2.

Lemma 8.3. *Assume that u is a solution to the obstacle problem in $B_1(0)$ and that, for some small ϵ,*

$$\|\nabla' u\|_{L^2(B_1(0))} = \epsilon.$$

Assume furthermore that $(e_1^1, e_2^1, ..., e_n^1)$ is a normalized coordinate system for u in $B_1(0)$ for some orthonormal vectors e_k^1, $k = 1, 2, ..., n$.

Then, possibly after relabeling the vectors $(e_1^1, e_2^1, ..., e_n^1)$, the angle between e_n^1 and e_n is estimates by

$$\arccos(e_n \cdot e_n^1) \leq C_n \epsilon, \tag{8.16}$$

or expressed diffrently

$$|e_n^1 \cdot e_n| \leq 1 - C_n \epsilon^2, \tag{8.17}$$

where the constant C_n only depends on n.

Proof: From the triangle inequality we may conclude that for any unit vector η

$$\|\eta \cdot \nabla u\|_{L^2(B_1(0)} \geq |\eta \cdot e_n| \left\| \frac{\partial u}{\partial x_n} \right\|_{L^2(B_1(0))} - \|\nabla' u\|_{L^2(B_1(0))} \tag{8.18}$$

We know, see Lemma 6.2, that if ϵ is small enough then $u \approx \frac{1}{2}(x_n)_+^2$. We may therefore assume that

$$\left\| \frac{\partial u}{\partial x_n} \right\|_{L^2(B_1(0))} = \left(\int_{B_1(0)} \left| \frac{\partial u}{\partial x_n} \right|^2 dx \right)^{1/2} \geq \frac{\sqrt{\omega_n}}{2}, \tag{8.19}$$

where ω_n is the volume of the unit ball.

Using (8.19) in (8.18) and that $\|\nabla' u\|_{L^2(B_1(0))}$ we can conclude that

$$\|\eta \cdot \nabla u\|_{L^2(B_1(0)} \geq |\eta \cdot e_n| \frac{\sqrt{\omega_n}}{2} - \epsilon. \tag{8.20}$$

If we choose a unit vector η in the span of $\{e_k^1;\ k = 1, 2, ..., n-1\}$ then we get

$$\epsilon = \|\nabla' u\|_{L^2(B_1(0))} \geq \|\eta \cdot \nabla u\|_{L^2(B_1(0)} \geq |\eta \cdot e_n| \frac{\sqrt{\omega_n}}{2} - \epsilon, \tag{8.21}$$

where we used that that the normalized coordinate system minimizes the norm in the first inequality. We can conclude that

$$|\eta \cdot e_n| \leq \frac{4\epsilon}{\sqrt{\omega_n}}. \tag{8.22}$$

Since $e_1^1, ..., e_n^1$ spans $\mathbb{R}^n$ we may write

$$e_n = (\eta \cdot e_n)\eta + \sqrt{1 - |\eta \cdot e_n|^2} e_n^1 \tag{8.23}$$

for some unit vector η in the span of $\{e_k^1;\ k = 1, 2, ..., n-1\}$. Taking the scalar product with e_n of both sides of (8.23) gives, after rearranging terms and simplifying,

$$|e_n^1 \cdot e_n| = \sqrt{1 - |\eta \cdot e_n|^2} = 1 - \frac{1}{2}|\eta \cdot e_n|^2 + O(|\eta \cdot e_n|^4) \leq 1 - \frac{16\epsilon^2}{\omega_n},$$

where we made a Taylor expansion in the second equality and used (8.22) together with the smallness assumption of ϵ in the inequality. The last estimate proves (8.17) and taking arccos of both sides in the last formula proves (8.16). □

The next Proposition corresponds to the main estimate (7.16) step 2 in the proof of Lemma 7.2. We choose to honor this technical result with the name of proposition since it is the heart of the iteration argument - as soon as we are able to derive an iterative estimate like (8.24) we are more or less done with the regularity proof.

Proposition 8.2. *For any $1/2 < \kappa < 1$ there exists an $\epsilon_\kappa > 0$ such that if u is a solution to the obstacle problem in $B_3(0)$ and*

1. *the coordinate system is normalized with respect to u*
2. *$0 \in \Gamma_u$,*
3. *$u(x) = 0$ in the set $\{x \in B_1(0);\ x_n < -1/2\}$ and that*
4. *$\|\nabla' u\|_{L^2(B_1(0))} = \epsilon < \epsilon_\kappa$.*

Then there exists coordinate systems $e_1^k, e_2^k, e_3^k, ..., e_n^k$ that are normalized with respect to the the functions

$$u_k(x) = u_{2^{-k}}(x) = 2^{2k} u(2^{-k}x).$$

Furthermore,

$$\left\|\nabla u_k - e_n^k(e_n^k \cdot \nabla u_k)\right\|_{L^2(B_1(0))} \leq \kappa^k \|\nabla' u\|_{L^2(B_1(0))} = \kappa^k \epsilon \tag{8.24}$$

and

$$\arccos(e_n^{k-1} \cdot e_n^k) \leq C_n \kappa^k \epsilon, \tag{8.25}$$

where C_n only depends on n.[35]

Proof: The proof follows from the previous analysis. We argue by induction. It follows from Proposition 8.1 that

$$\|\nabla' u\|_{L^2(B_{1/2}(0))} \leq \frac{\kappa}{2^{\frac{n+2}{2}}} \|\nabla' u\|_{L^2(B_1(0))}. \tag{8.26}$$

Notice that, by a change of variables $x \mapsto x/2$ this means that

$$\int_{B_{1/2}(0)} |\nabla' u|^2 dx = \frac{1}{2^{n+2}} \int_{B_1(0)} \Big|\nabla' \underbrace{\frac{u(2^{-1}x)}{2^{-2}}}_{=u_1}\Big|^2 dx = \tag{8.27}$$

$$= \frac{1}{2^{n+2}} \|\nabla' u_1\|^2_{L^2(B_1(0))}.$$

[35] We interpret $e_n^0 = e_n$ in this Proposition.

By the minimality in the assumption of normalized coordinate system and (8.27), and then (8.26), it follows that

$$\frac{1}{2^{\frac{n+2}{2}}}\left\|\nabla u_1 - e_n^1(e_n^1\cdot\nabla u_1)\right\|_{L^2(B_1(0))} \le \|\nabla' u\|_{L^2(B_{1/2}(0))} \le \tag{8.28}$$

$$\le \frac{\kappa}{2^{\frac{n+2}{2}}}\|\nabla' u\|_{L^2(B_1(0))} = \frac{\kappa}{2^{\frac{n+2}{2}}}\epsilon.$$

This means that (8.24) holds for $k=1$ and therefore, by Lemma 8.3, that (8.25) holds for $k=1$.

Next we assume that (8.24) and (8.25) holds for an arbitrary $k\ge 1$ then $u_k(x)$ satisfies the assumptions of Proposition 8.1, in the coordinate system $e_1^k, e_2^k, e_3^k, ..., e_n^k$ with $\frac{\kappa^k}{2^k}\epsilon < \epsilon$ in place of ϵ. We may conclude, from an argument like (8.26)-(8.28) (applied to u_k instead of u) that (8.24) holds for $k+1$; and we may therefore apply Lemma 8.3 to u_{k+1} and conclude that (8.25) holds for $k+1$. The proposition follows by induction. □

We are now ready to prove that the free boundary have an normal at the origin.

Corollary 8.1. *Assume that u satisfies the assumptions of Proposition 8.2. Then the limit $\lim_{k\to\infty} e_n^k = \nu(0)$ exists and*

$$|e_n - \nu(0)| \le C_n\epsilon \tag{8.29}$$

where C_n only depends on n.

Proof: Notice that the angle between the approximate normals e_n^j and e_n^l, for $l>j$, can be estimated

$$\arccos(e_n^j\cdot e_n^l) \le \sum_{k=j+1}^{l}\arccos(e_n^{k-1}\cdot e_n^k) \le C_n\sum_{k=j+1}^{l}\kappa^k\epsilon \le$$

$$\le C_n\frac{\kappa^{j+1}}{(1-\kappa)}\epsilon \le C_n\kappa^j\epsilon.$$

We can conclude that

$$|e_n^j - e_n^l|^2 = 2 - 2\cos\left(C_n\kappa^j\epsilon\right) \le C_n^2\left(\kappa^j\epsilon\right)^2, \tag{8.30}$$

where we used the elementary inequality $\cos(\alpha)\ge 1-\frac{1}{2}\alpha^2$.

It follows from (8.30) that the sequence e_n^j forms a Cauchy sequence and is thus convergent to some vector $\nu(0)$. The estimate (8.29) follows from (8.30) by choosing $l=0$ and sending $j\to\infty$. □

Finally we can prove the result corresponding to Step 3 in the proof of Lemma 7.2.

Corollary 8.2. *The vector $\nu(0)$ in Corollary 8.1 is the measure theoretic normal of Γ_u at the origin and furthermore*

$$\|\nabla u_k - \nu(0)(\nu(0)\cdot\nabla u_k)\|_{L^2(B_1(0))} \le C_n\kappa^j\epsilon, \tag{8.31}$$

where C_n only depends on the dimension n.

Proof: We begin by showing (8.31). This estimate this we begin to apply the triangle inequality

$$\|\nabla u_k - \nu(\nu\cdot\nabla u_k)\|_{L^2(B_1(0))} \le \|\nabla u_k - e_n^k(e_n^k\cdot\nabla u_k)\|_{L^2(B_1(0))}+$$
$$+\|\nu(\nu\cdot\nabla u_k) - e_n^k(e_n^k\cdot\nabla u_k)\|_{L^2(B_1(0))} \le \tag{8.32}$$
$$\le \kappa^k\epsilon + \|\nu(\nu\cdot\nabla u_k) - e_n^k(e_n^k\cdot\nabla u_k)\|_{L^2(B_1(0))},$$

where we used (8.24) in the last inequality.

To estimate the last term in (8.32) we use (8.30)

$$\|\nu(\nu\cdot\nabla u_k) - e_n^k(e_n^k\cdot\nabla u_k)\|_{L^2(B_1(0))} =$$
$$= \|\nu((\nu - e_n^k)\cdot\nabla u_k) + (\nu - e_n^k)(e_n^k\cdot\nabla u_k)\|_{L^2(B_1(0))} \le$$
$$\le C_n\left(\kappa^k\epsilon\right)\|\nabla u_k\|_{L^2(B_1(0))} \le C_n\kappa^k\epsilon,$$

where we included $\|\nabla u_k\|_{L^2(B_1(0))}$ into the constant in the last step.[36]

The conclusion of the Corollary follows from this last estimate and (8.32). □

Theorem 8.1. *For any $0<\alpha<1$ there exists an $\epsilon_\alpha>0$ such that if u is a solution to the obstacle problem in $B_3(0)$ and*

1. *the coordinate system is normalized with respect to u*
2. *$0\in\Gamma_u$,*
3. *$u(x)=0$ in the set $\{x\in B_1(0);\ x_n<1/2\}$ and that*
4. *$\|\nabla' u\|_{L^2(B_1(0))}=\epsilon<\epsilon_\alpha$.*

Then the free boundary $\Gamma_u=\{(x',f(x'));x'\in B'_{1/2}(0)\}$, in $B_{1/2}(0)$, where $f(x')$ is a $C^{1,\alpha}(B'_{1/2}(0))$ function. Furthermore, the function f satisfies the following estimate

$$\|f\|_{C^{1,\alpha}(B'_{1/2}(0))}\le C_n\epsilon, \tag{8.33}$$

where C_n only depend on the dimension.

Proof: We will prove the Theorem in several steps. First we fix an $0<\alpha<1$ and choose a $1/2<\kappa<1$, we will specify how κ depend on α at the end of the proof. Let$\epsilon_\kappa>0$ be as in Proposition 8.2 and choose $\epsilon_\alpha=2^{-(n+2)}\epsilon_\kappa$.

Step 1: *If u is as in the Theorem with $\epsilon<\epsilon_\alpha$ then the measure theoretic normal of Γ_u is well defined at any point $y\in\Gamma_u\cap B_{1/2}(0)$.*

Proof of step 1. If $y\in\Gamma_u\cap B_{1/2}(0)$ then the rescaled function

$$u_{2^{-1},y}(x)=\frac{u(2^{-1}x+y)}{2^{-2}}$$

[36]Notice that $\|\nabla u_k\|_{L^2(B_1(0))}$ only depend on the dimension since $u_k\approx\frac{1}{2}(x_n)_+^2$.

will be a solution to the obstacle problem in $B_1(0)$ and

$$\int_{B_1(0)} |\nabla' u_{2^{-1},y}(x)|^2 dx = 2^{n+2} \int_{B_{1/2}(y)} |\nabla' u(x)|^2 dx \leq \tag{8.34}$$

$$\leq 2^{n+2} \int_{B_1(0)} |\nabla' u(x)|^2 dx \leq 2^{n+2}\epsilon^2 < 2^{-(n+2)}\epsilon_\kappa^2,$$

where the first inequality follows since $B_{1/2}(y) \subset B_1(0)$.

We may therefore find a normalized coordinate system $e_1^y, ..., e_n^y$ such that the assumptions of Proposition 8.2 holds for $u_{2^{-1},y}(x)$ in $B_1(0)$. The only assumption that needs to be checked is that $u_{2^{-1},y}(x) = 0$ in the set $x \cdot e_n^y < 1/2$; but that follows if ϵ is small enough since then Ω_u is close to the set $\{x;\ x_n < 0\}$ by Lemma 6.2. The existence of the measure theoretic normal is therefore assured by Proposition 8.2.

Step 2: *Let $\nu(y)$ be the measure theoretic normal at the point $y \in \Gamma_u \cap B_{1/2}(0)$. Then*

$$|\nu(0) - \nu(y)| \leq C_n \kappa^k \epsilon \tag{8.35}$$

where C_n only depend on the dimension.

Furthermore, if $\kappa \leq 2^{-\alpha}$ then

$$|\nu(0) - \nu(y)| \leq C_n \epsilon |y|^\alpha, \tag{8.36}$$

where C_n only depend on the dimension.

Proof of step 2. Let k be the smallest natural number, $k = 0, 1, 2, ...$, such that $2^{-k-2} \leq |y| < 2^{-k-1}$. Then, by Corollary 8.2,

$$\|\nabla u_k - \nu(0)(\nu(0) \cdot \nabla u_k)\|_{L^2(B_1(0))} \leq C_n \kappa^k \epsilon. \tag{8.37}$$

Furthermore, the point $y \in \Gamma_u$ will be moved to $2^k y \in \Gamma_{u_k}$ by the dilation $x \mapsto 2^k$.

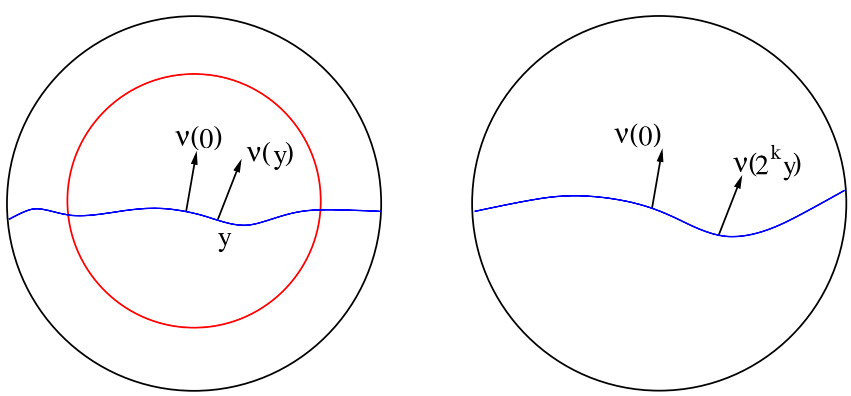


Figure: *In the graph we indicate the geometry of the argument. In the left picture we have the point $y \in \Gamma_u$ with normal $\nu(y)$. But if we rescale the red ball $B_{2^{-k}}(0)$ to the unit ball $B_1(0)$ (in the left picture) by the dilation $x \mapsto 2^k$ then y maps to $2^k y$. Notice that the right rescaling of $u(x)$ under the dilation is $u(x) \mapsto u_k(x) = u_{2^{-k}}(x) = 2^{2k}u(2^{-k}x)$.*

Arguing as in (8.34) we may estimate

$$\int_{B_1(0)} \left|\nabla u_{2^{-k-1},y}(x) - \nu(0)(\nu(0)\cdot\nabla u_{2^{-k-1},y}(x))\right|^2 dx =$$
$$= 2^{n+2}\int_{B_{1/2}(2^k y)} \left|\nabla u_k(x) - \nu(0)(\nu(0)\cdot\nabla u_k)\right|^2 dx \leq \tag{8.38}$$
$$\leq 2^{n+2}\int_{B_1(0)} \left|\nabla u_k(x) - \nu(0)(\nu(0)\cdot\nabla u_k)\right|^2 dx \leq \left(C_n\kappa^k\epsilon\right)^2.$$

From Lemma 8.3 it follows that the coordinate system $\hat{e}_1, \hat{e}_2, ..., \hat{e}_n$ normalized with respect to $u_{2^{-k-1},y}(x)$ is rotated by an angle that can be controlled as follows

$$\arccos(\hat{e}_n \cdot \nu(0)) \leq C_n\frac{\kappa^k}{2^k}\epsilon \Rightarrow |\hat{e}_n - \nu(0)| \leq C_n\kappa^k\epsilon. \tag{8.39}$$

From Corollary 8.1 and (8.38) we can conclude that

$$|\nu(y) - \hat{e}_n| \leq C_n\kappa^k\epsilon \tag{8.40}$$

From (8.39) and (8.40) together with the triangle inequality we can estimate

$$|\nu(y) - \nu(0)| \leq |\hat{e}_n - \nu(0)| + |\nu(y) - \hat{e}_n| \leq C_n\kappa^k\epsilon, \tag{8.41}$$

this proves (8.35).

To prove (8.36) we just use that if $\kappa \leq 2^{-\alpha}$, which implies that

$$|y|^\alpha \geq |2^{-k-2}|^\alpha \geq |2^{-k}|^\alpha \geq \kappa^k, \tag{8.42}$$

where we used that $|y| \geq 2^{-k-2}$ by the choice of k. Using (8.42) in (8.41) we arrive at:

$$|\nu(y) - \nu(0)| \leq |\hat{e}_n - \nu(0)| + |\nu(y) - \hat{e}_n| \leq C_n\kappa^k\epsilon \leq C_n|y|^\alpha\epsilon$$

which proves (8.36).

Step 3: *The normal of the free boundary, $\nu(x)$, satisfies*

$$|\nu(y) - \nu(z)| \leq C_n\epsilon|y-z|^\alpha \quad \text{for all } y, z \in \Gamma_u \cap B_{1/2}(0). \tag{8.43}$$

Proof of Step 3: By the estimate in step 2 it is enough to prove (8.43) for points $|y - z| < 1/4$.[37]

[37] If $|y-z| \geq 1/4$ then $|\nu(y) - \nu(z)| \leq |\nu(0) - \nu(y)| + |\nu(0) - \nu(z)| < C_n\epsilon(|y|^\alpha + |z|^\alpha) \leq C_n\epsilon|y-z|^\alpha$.

The proof of this part just by noticing that if $y, z \in B_{1/2}(0)$ then, by (8.34),

$$\int_{B_1(0)} |\nabla' u_{2^{-1},y}(x)|^2 dx = 2^{n+2} \int_{B_{1/2}(y)} |\nabla' u(x)|^2 dx < 2^{-(n+2)} \epsilon_\kappa^2.$$

We may therefore apply exactly the same argument as in step 2 to the function $u_{2^{-1},y}(x)$ for any $z \in \Gamma_u \cap B_{1/4}(y)$.

The proof is now easy to finish. By step 3 the normal of the free boundary is Hölder continuous, this means that the free boundary can be written as a graph of a $C^{1,\alpha}-$function $f(x')$. By Corollary 8.1 it follows that $|\nabla' f(x')| \leq C_n \epsilon$ and by assumption $f(0) = 0$. Clearly these last facts implies (8.33). □

Corollary 8.3. *Let u be a solution to the obstacle problem in $\mathcal{D}$ and $\alpha < 1$. Then, for $\mathcal{H}^{n-1}-$a.e. free boundary point $z \in \Gamma_u$, there exists a ball $B_{r_z}(z)$, for some $r_z > 0$, such that $\Gamma_u \cap B_{r_z}(z)$ is a $C^{1,\alpha}-$graph.*

Proof: By Theorem 6.1, Theorem 6.2 and Lemma 6.1 we know that for a.e. free boundary point z

$$\lim_{r \to 0} \frac{u(rx+z)}{r^2} = \frac{1}{2}(\nu(z) \cdot x)_+^2$$

for some unit normal $\nu(z)$. We may assume, by rotating the coordinate system, that $\nu(z) = e_n$. It follows that, for r small enough (say $r = 2r_z$),

$$\left\| \nabla' \frac{u(rx+z)}{r^2} \right\|_{L^2(B_1(0))} < \epsilon_\alpha,$$

where ϵ is as in Theorem 8.1.

From Theorem 8.1 we may therefore conclude that the free boundary of $\frac{u(rx+z)}{r^2}$ is a $C^{1,\alpha}-$graph in $B_{1/2}(0)$. But this is the same as the free boundary $\Gamma_u \cap B_{r/2}(z) = \Gamma_u \cap B_{r_z}(z)$ is a $C^{1,\alpha}-$graph. □